Group Seminar / Dipartimento di Fisica - Tor Vergata - 23/09/2022

Turbulent Circulation Statistics: Recent Advances and Perspectives

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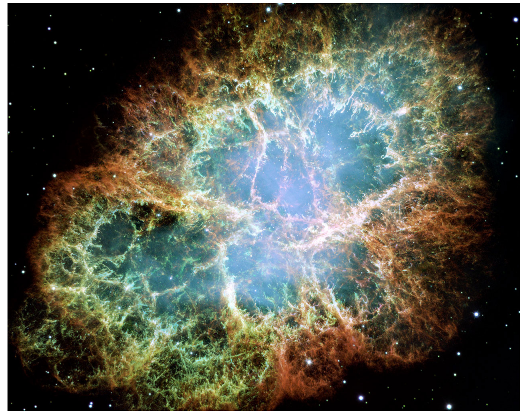
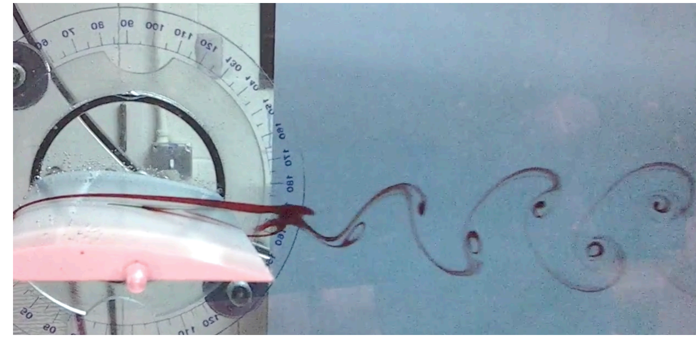
Circulation is all Around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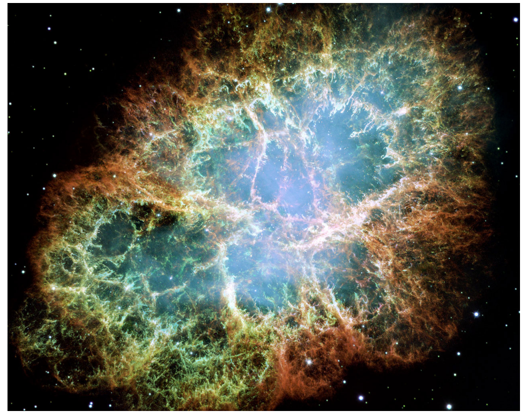

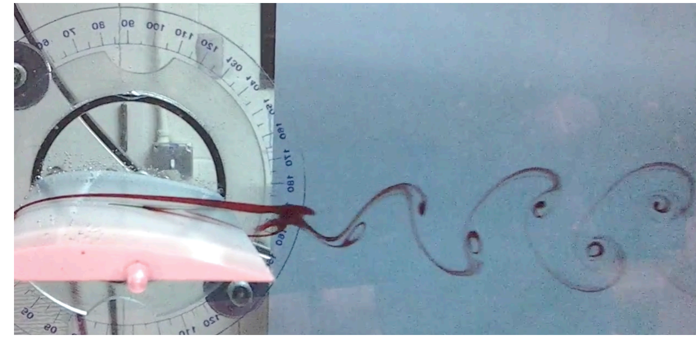


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Talk Outline

I. Homogeneous and Isotropic Turbulence

II. Circulation Statistics

III. Vortical Structures & Multiplicative Cascades

IV. Multifractality Breaking

V. Extreme Circulation Events

VI. Circulation Statistics & Minimal Surfaces

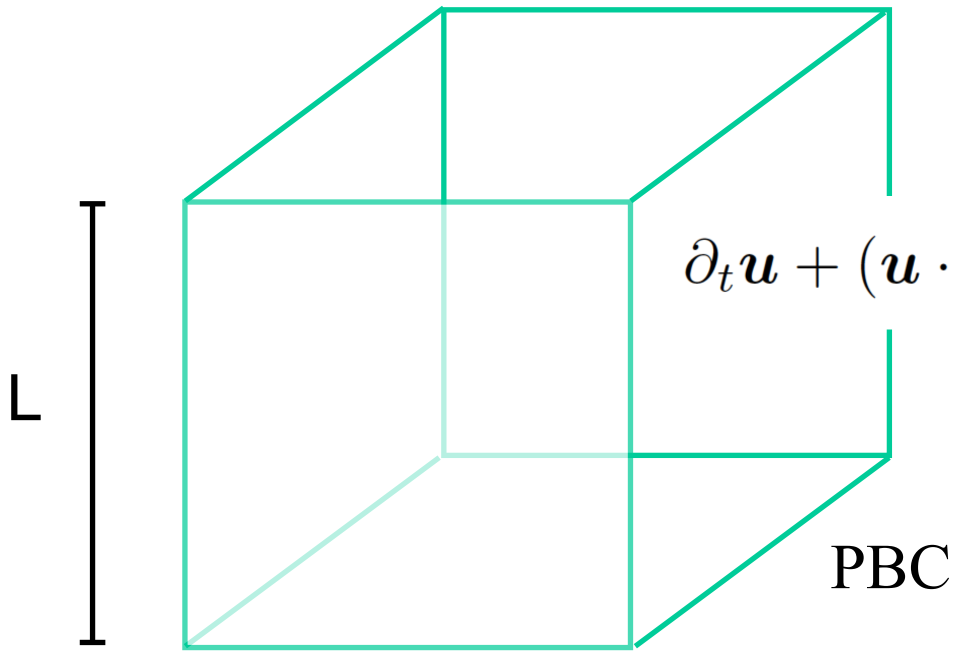
VII. Conclusions & Outlook

I. Homogeneous and Isotropic Turbulence

Navier-Stokes Equations

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{u} + \boldsymbol{f}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{u}(\boldsymbol{x},t) = 0.$$

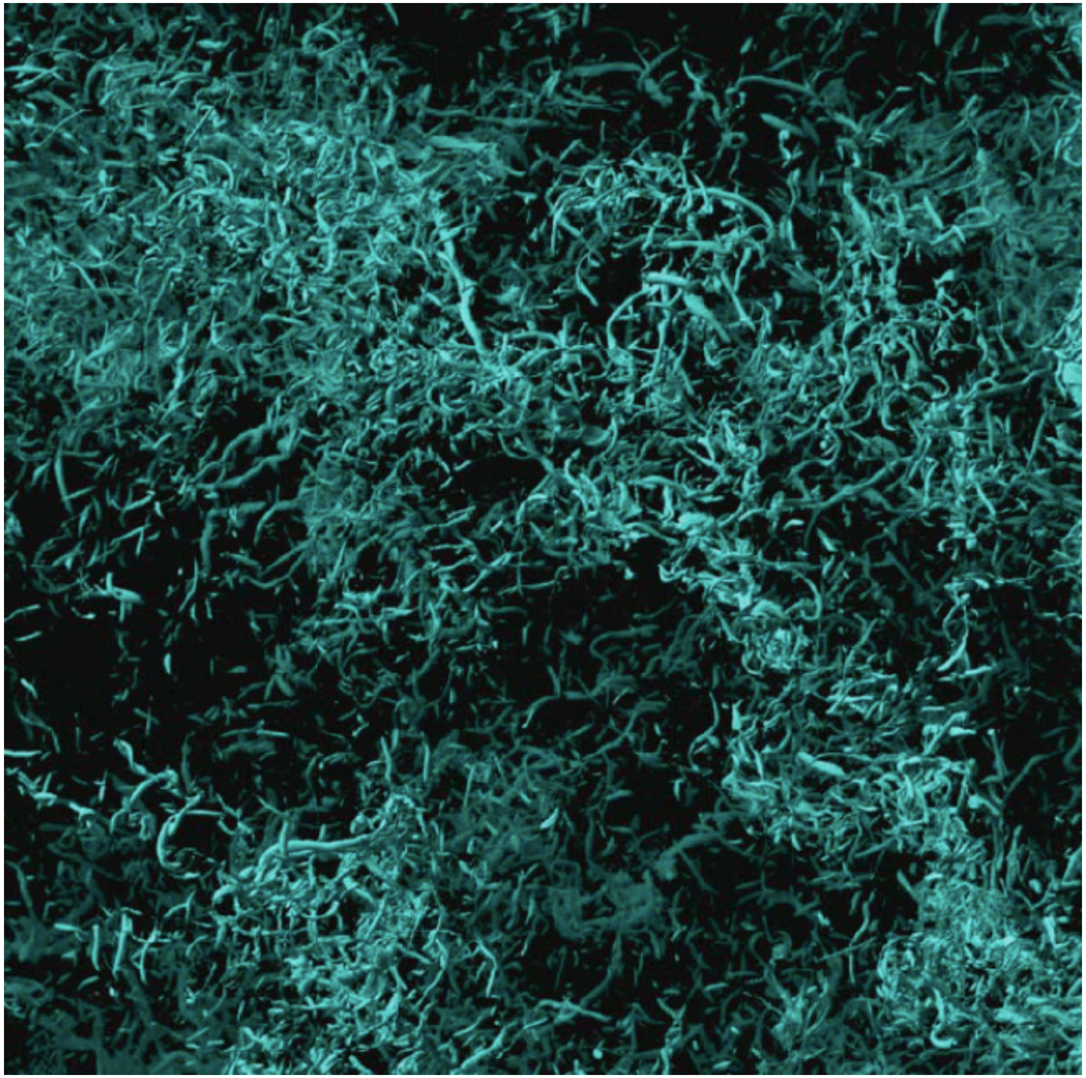
$$\boldsymbol{x} \to L\boldsymbol{x}, \quad \boldsymbol{u} \to U\boldsymbol{u}, \quad t \to \frac{L}{U}t, \quad \boldsymbol{f} \to \frac{U^2}{L}\boldsymbol{f}, \quad p \to U^2 p,$$



$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + \frac{1}{\mathrm{Re}} \boldsymbol{\nabla}^2 \boldsymbol{u} + \boldsymbol{f}$$

$$\mathrm{Re} = LU/\nu$$

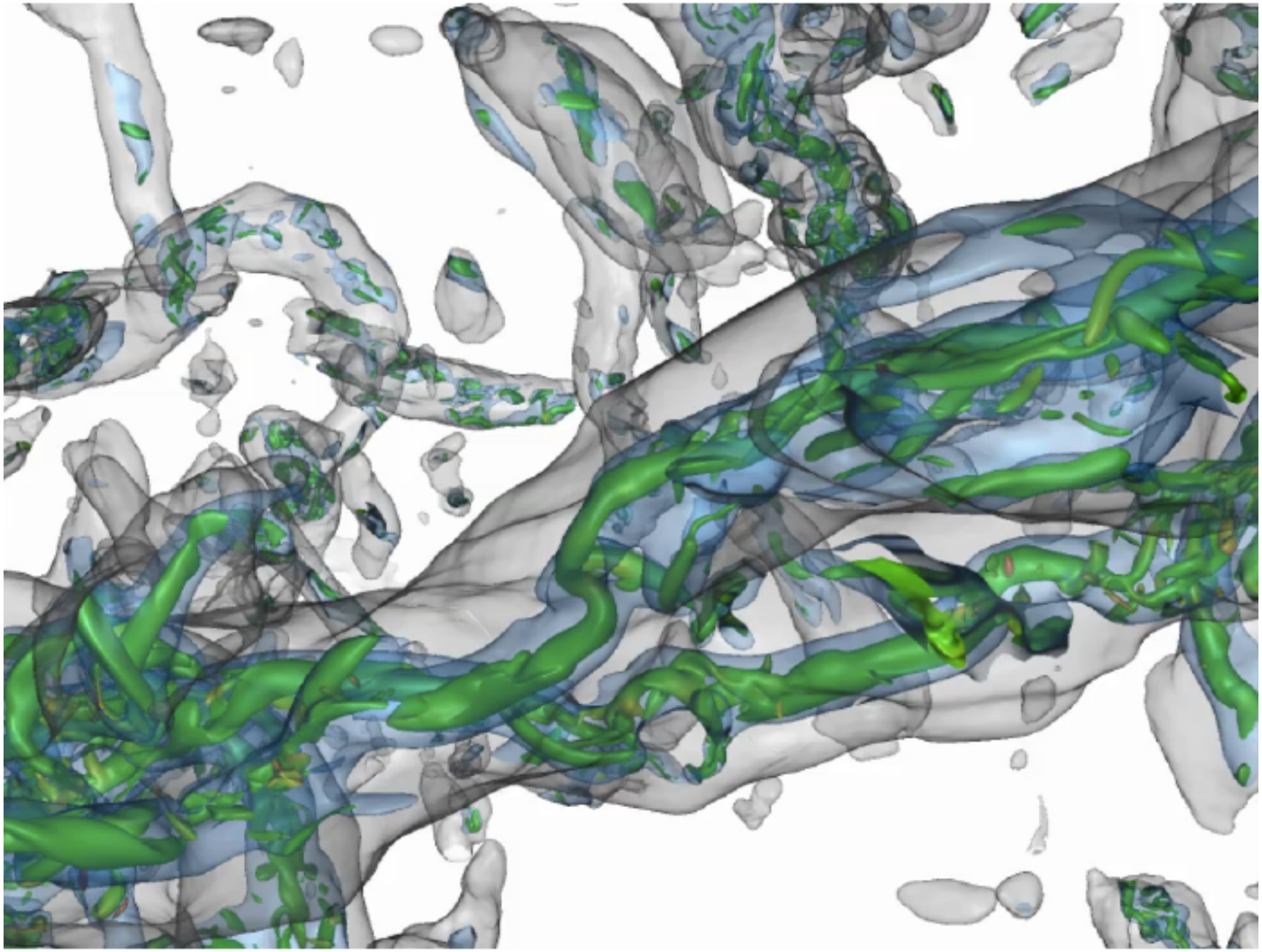
This is what we see in turbulence - Part I of II



High vorticity regions

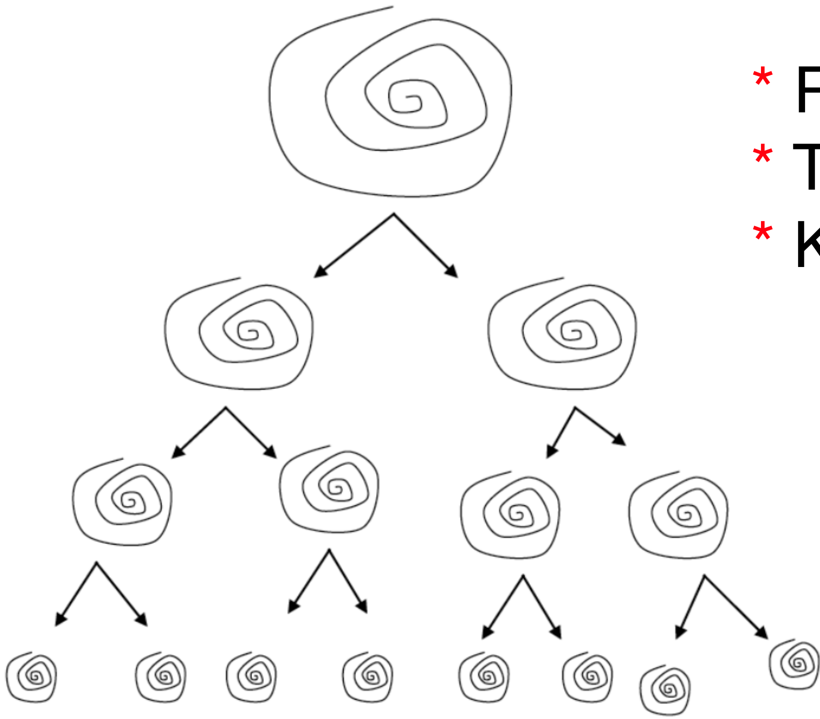
Kaneda and Ishihara JT (2006) - Earth Simulator

This is what we see in turbulence - Part II of II

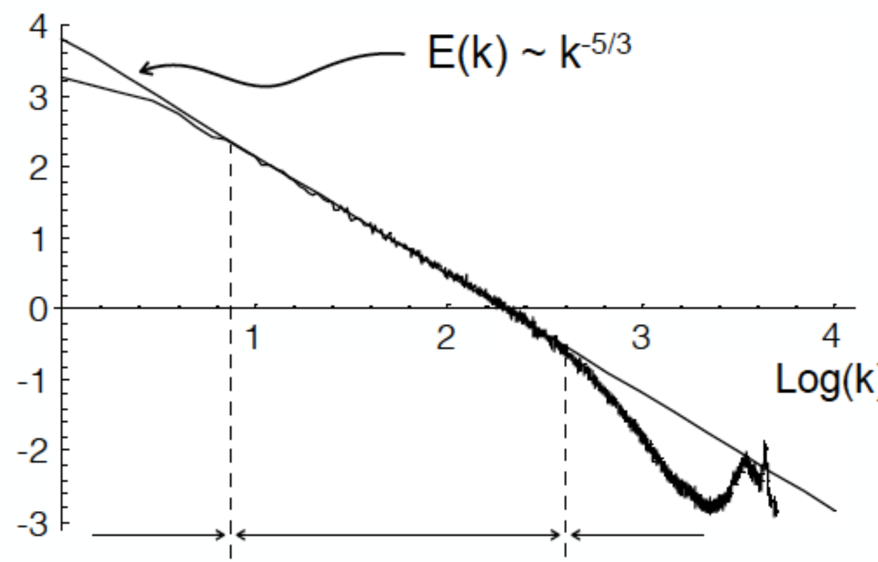


"Vortices within vortices"
Bürger et al. ArXiv (2012)

Turbulent Cascade : Energy Flows from Large to Small Scales
"K41 Phenomenology"



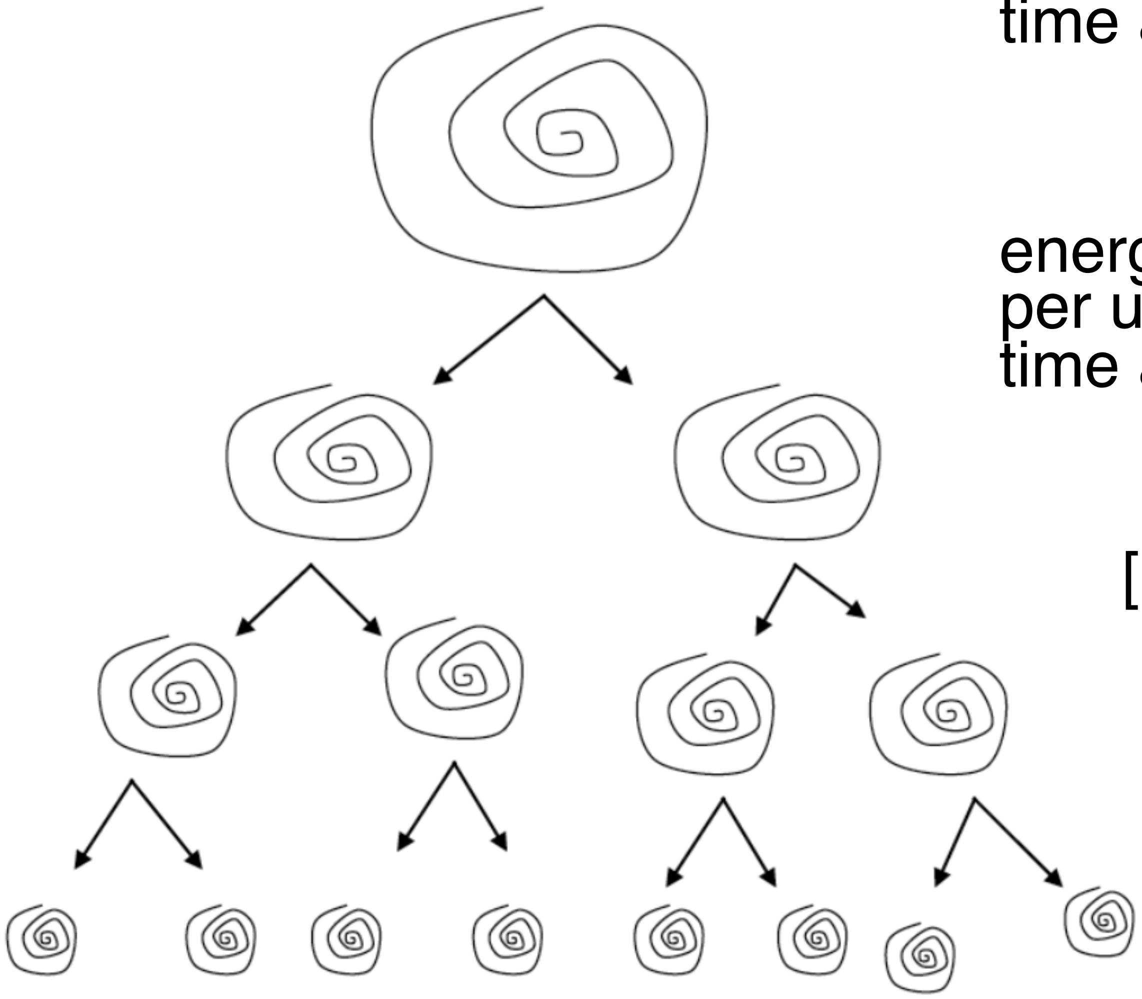
* Richardson (1922)
* Taylor (1935)
* Kolmogorov (1941)



Big whirls have little whirls
that feed on their velocity,
and little whirls have lesser whirls
and so on to viscosity
— in the molecular sense.

L >> R >> $\eta_K = (\nu^3/\epsilon)^{1/4}$

L : integral scale



ε : energy injected per unit mass per unit time at large scales

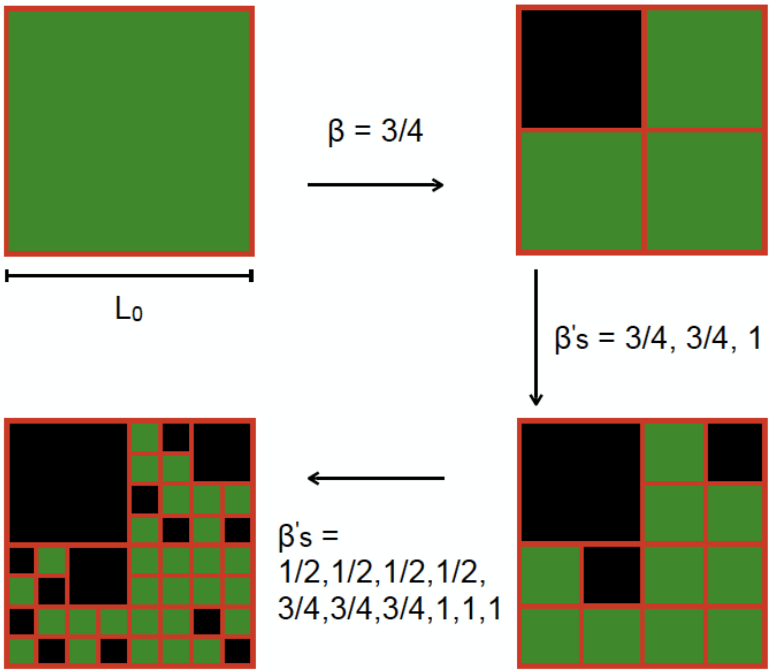
=

energy dissipated per unit mass per unit time at small scales

$[\varepsilon] = L^2 / T^3$

η_K : dissipative scale

Intermittency Modeling



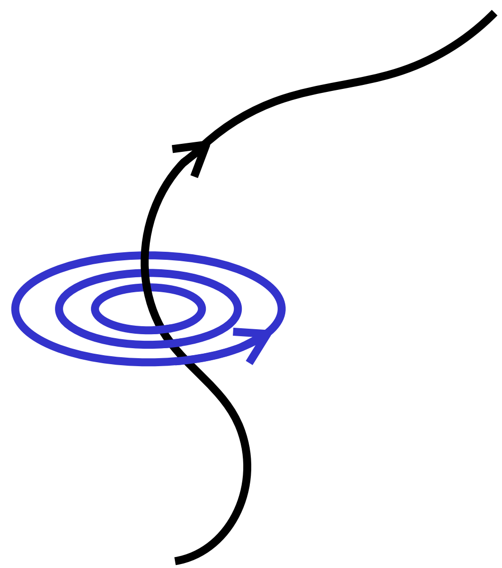
Multiplicative Cascades

* Obukhov, Kolmogorov JFM (1962)
"OK62"
* Mandelbrot JFM (1974)
* Frisch et al. JFM (1978)
* Benzi et al. JPA (1984)

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{4\pi}\int d^3x' \frac{\boldsymbol{\omega}(\boldsymbol{x}',t)\times(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}$$

Structural Approach

Farge et al. PRL (2004)



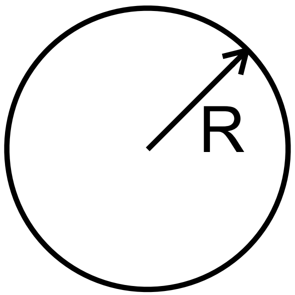
II. Circulation Statistics

A. Migdal [Int. J. Mod. Phys. A (1994)], inspired by the Wilson Loop,

$$W = \mathrm{Tr}\,(\,\mathcal{P} \exp i \oint_C A_\mu dx^\mu\,)$$

approach to the confinement problem in QCD, introduced in 1994 an analytical framework to model turbulence, based on the circulation variable

$$\Gamma_R \equiv \int_{\mathcal{D}} d^2\mathbf{r}\, \omega(\mathbf{r}) \quad .$$



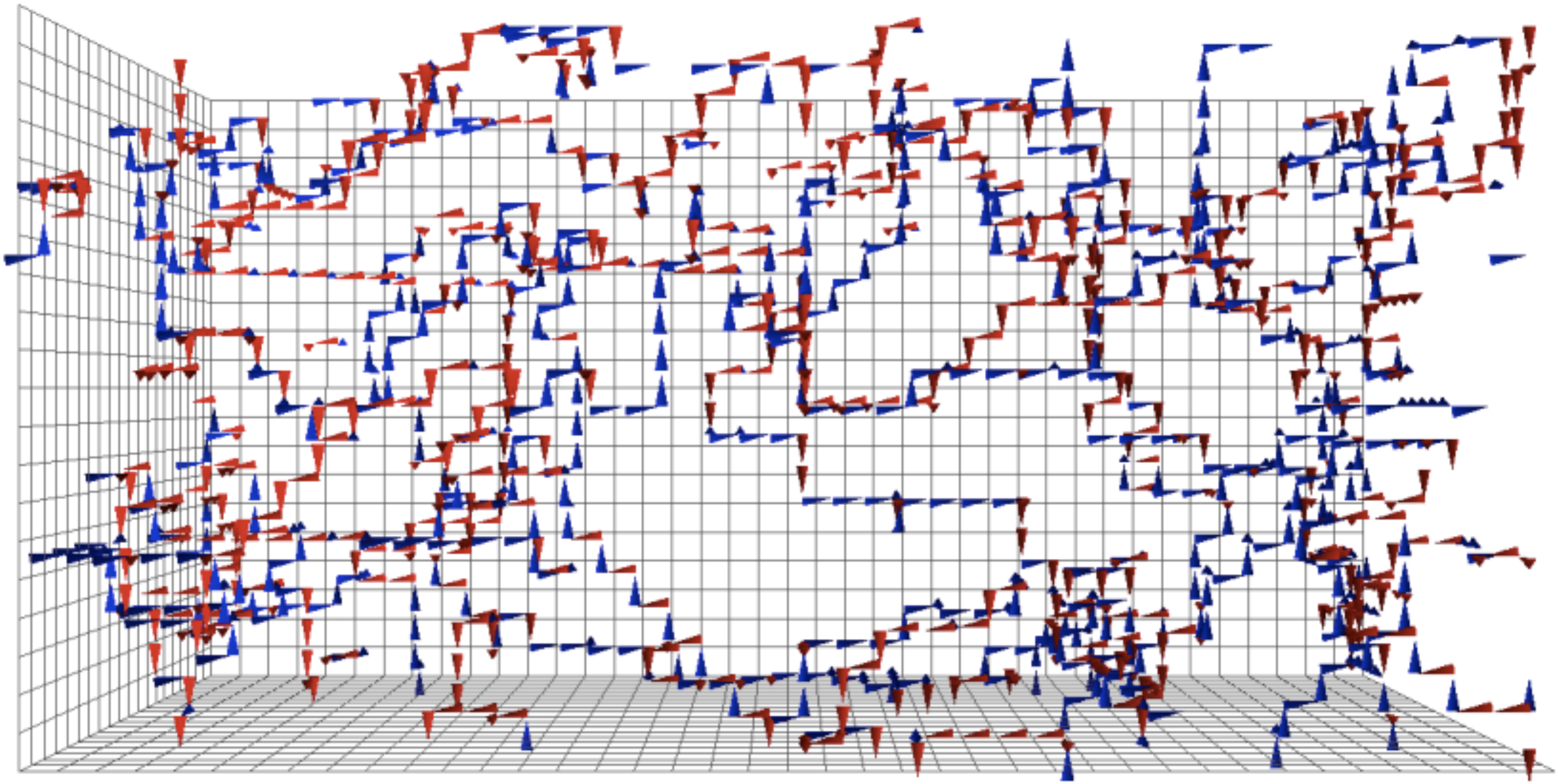
Time is now ripe for extensive numerical simulations.

Iyer, Sreenivasan and Yeung - PRX (2019)

Lattice size 16384^3 with $R_\lambda = 1300$

* Texas Advanced Computing Center - Texas University (Austin)
* Blue Project - University of Illinois (Urbana-Champaign)

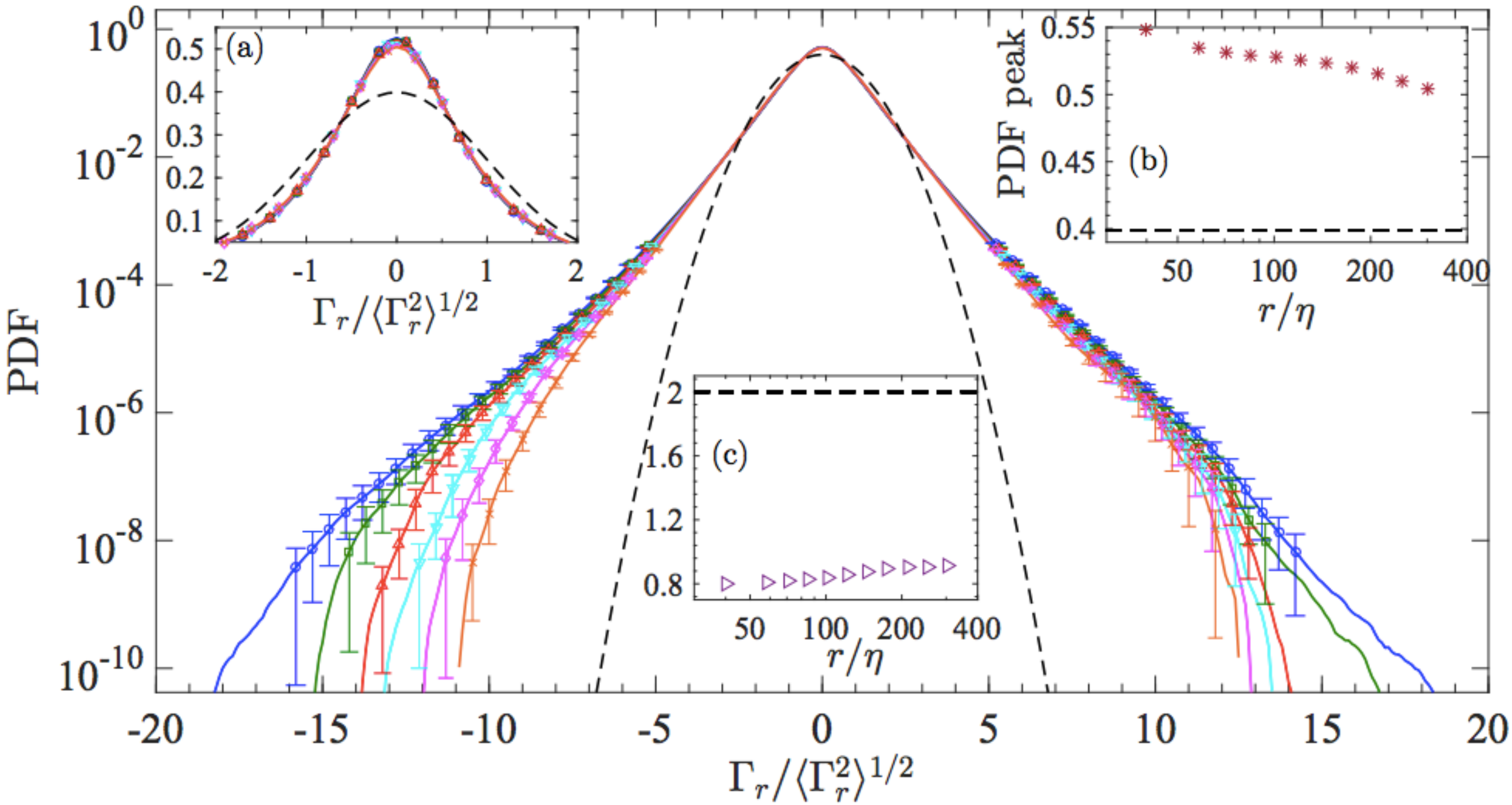
A pictorial digression on the QCD center vortices



Biddle, Kamleh, and Leinweber PRD (2020)

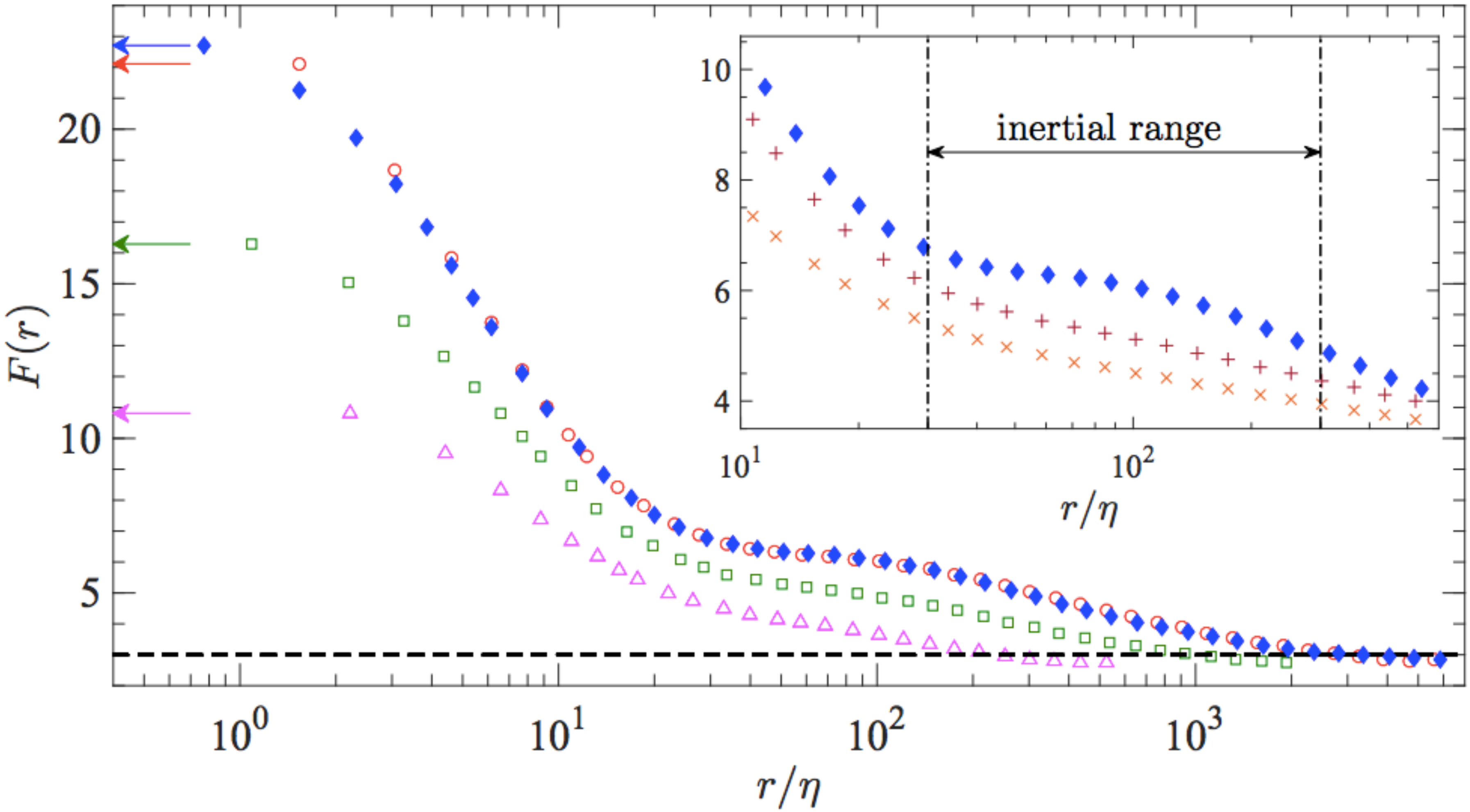
Iyer, Sreenivasan and Yeung - PRX (2019)

Circulation PDFs



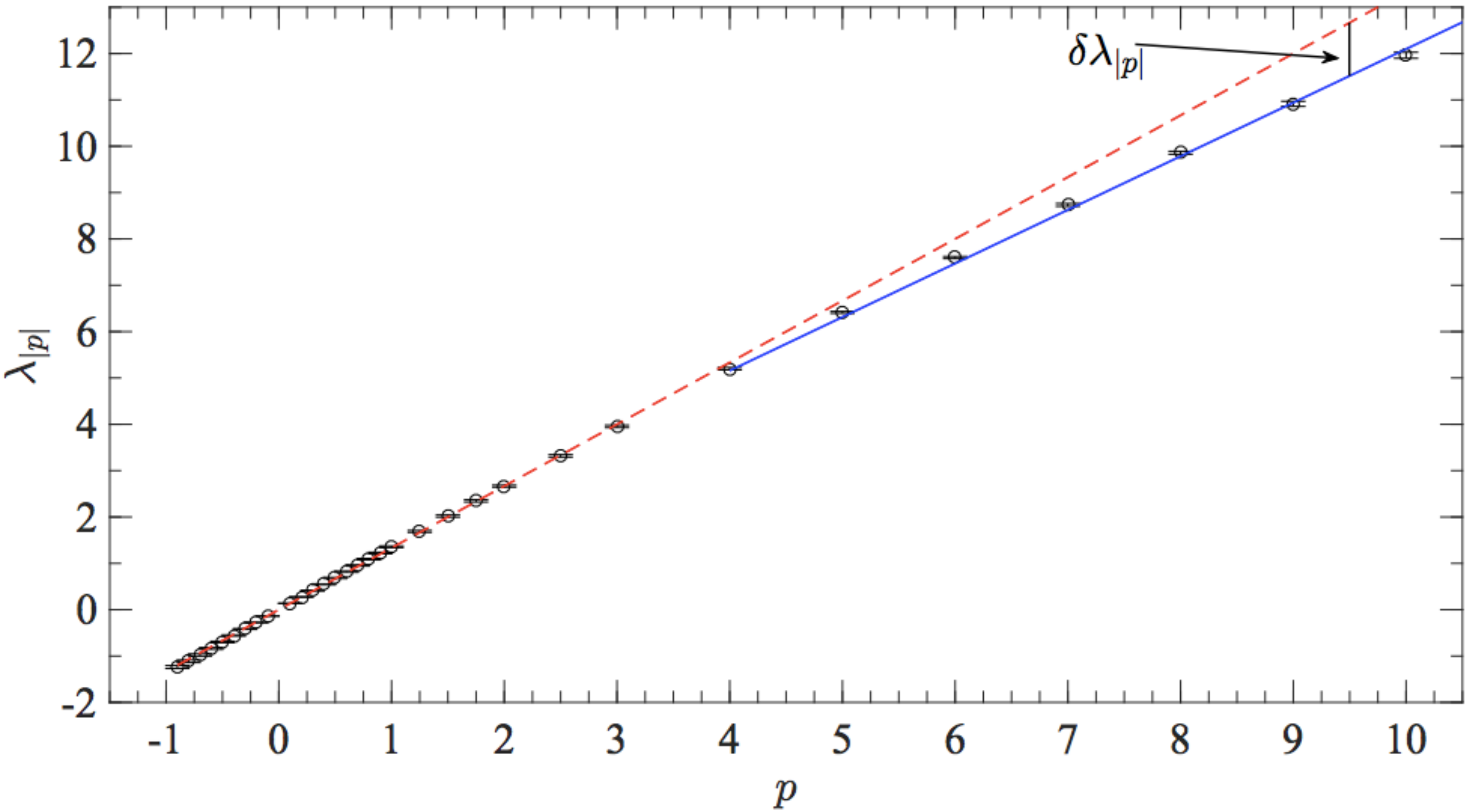
Iyer, Sreenivasan and Yeung - PRX (2019)

Circulation Kurtoses $F(r) = \mathbb{E}[\Gamma^4]/(\mathbb{E}[\Gamma^2])^2$



Iyer, Sreenivasan and Yeung - PRX (2019)

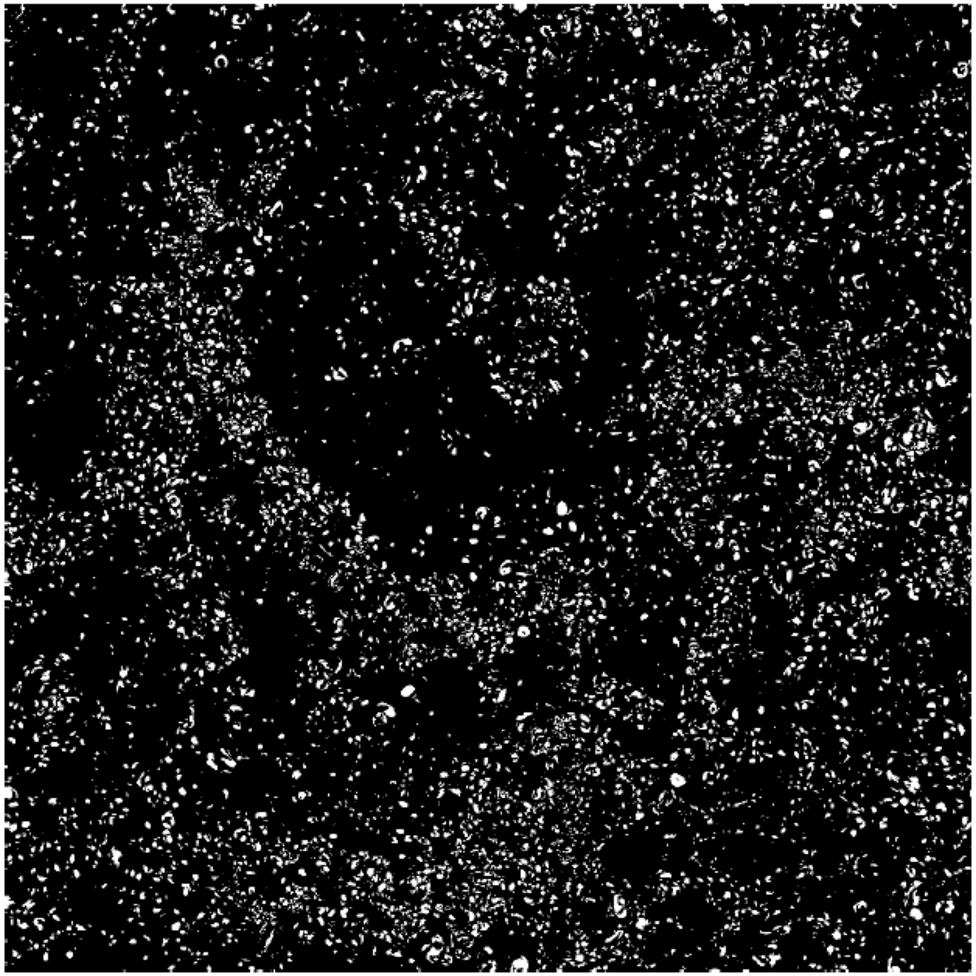
Scaling exponents of circulation moments $\mathbb{E}[\Gamma^q] \sim r^{\lambda_q}$

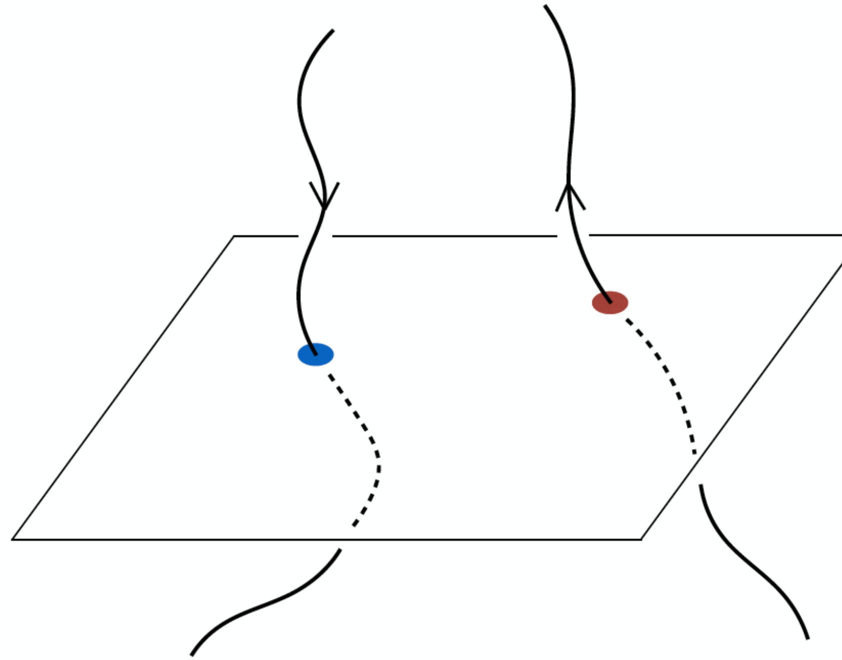


III. Vortical Structures & Multiplicative Cascades

Vortex spots in a slicing plane

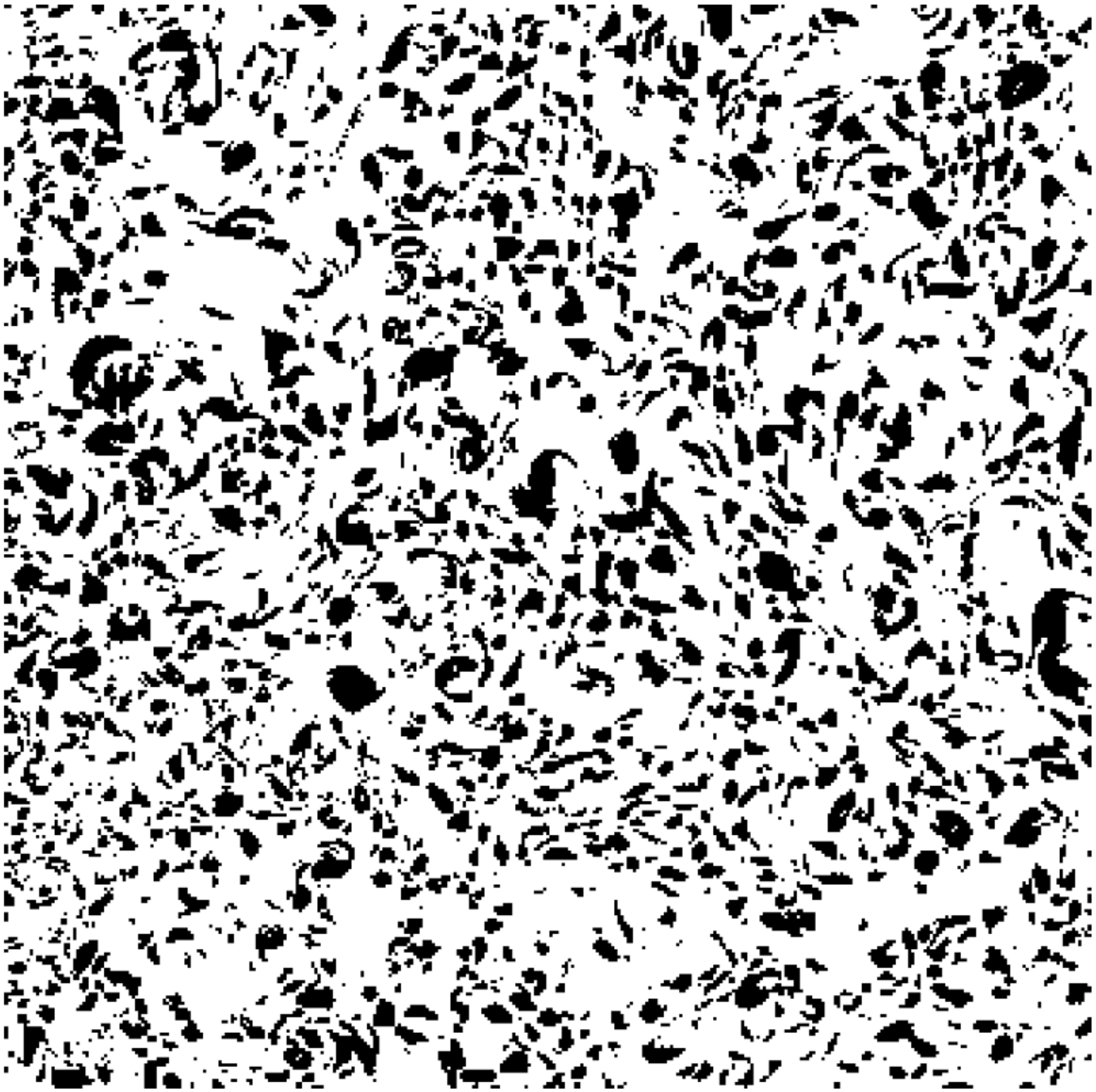
Raw simulation data from the Johns Hopkins University Turbulence Database (JHTD)





1024 x 1204

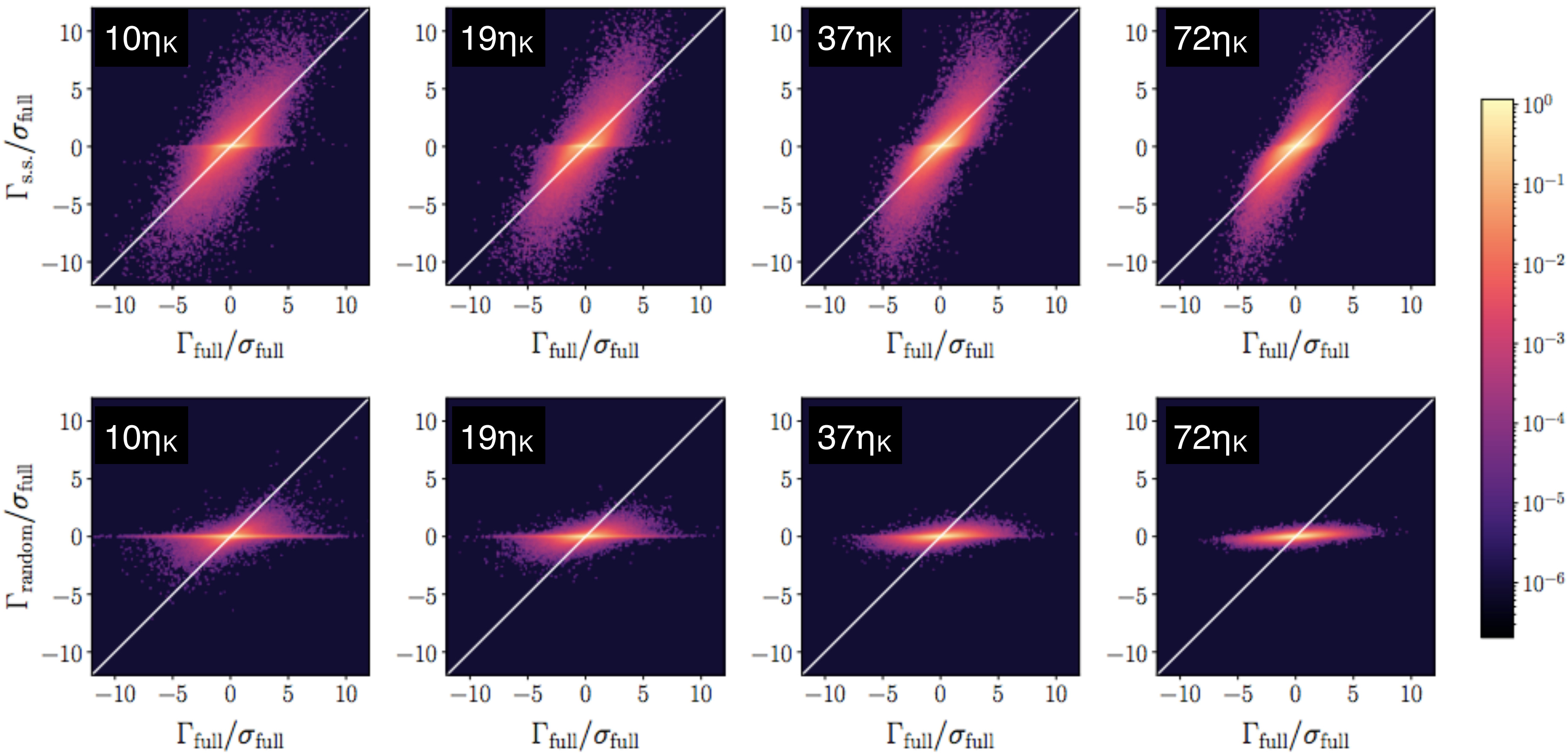
G.B. Apolinário, L.M., R.M. Pereira, and V.J. Valadão, PRE RC (2020)



350 x 350 (JHTD)

Do the spots saturate circulation?

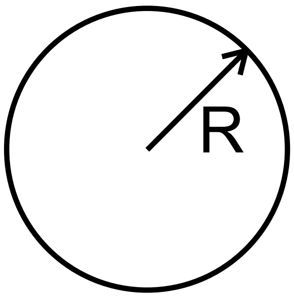
Data (JHTD) for $R_\lambda = 610$:



L.M., R.M. Pereira, and V.J. Valadão, PRE Letter (2022).

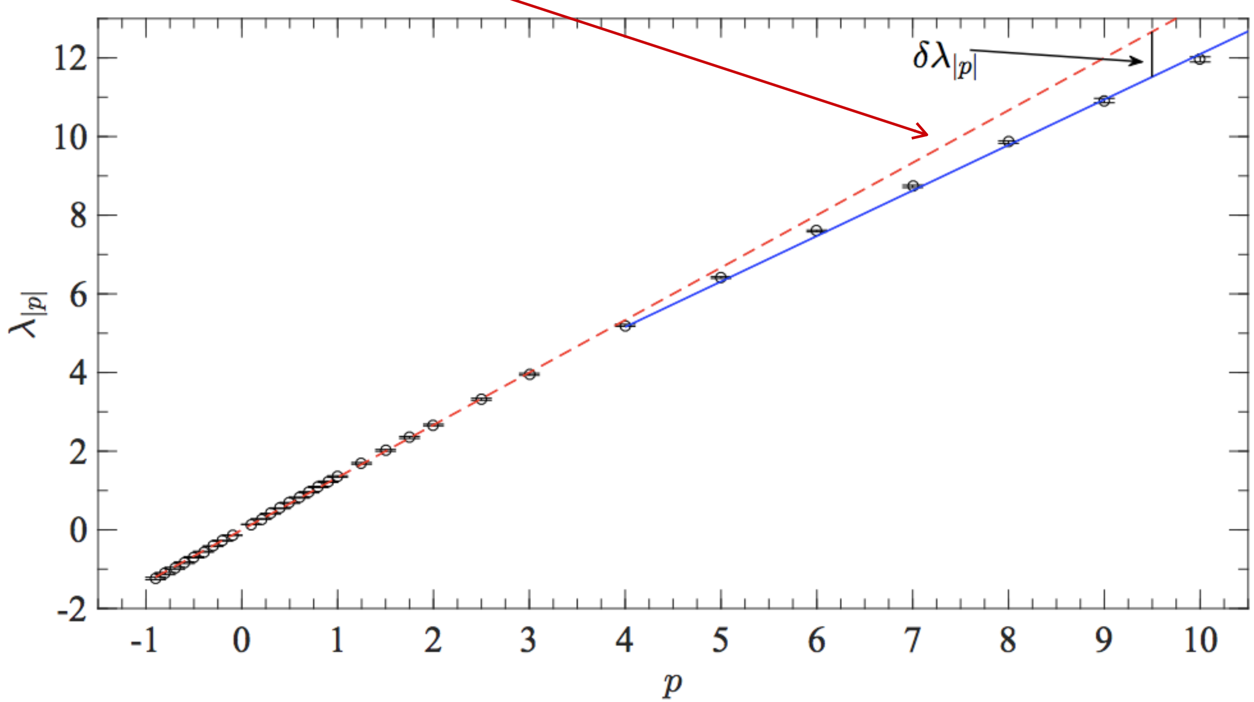
Circulation sounds to be a suitable observable for the fusion of multiplicative cascade and structural ideas.

$$\Gamma_R \equiv \int_{\mathcal{D}} d^2\mathbf{r}\, \omega(\mathbf{r})$$



Disc of radius R

K41: $\langle \Gamma_R^p \rangle \sim \epsilon^{\frac{p}{3}} R^{\frac{4p}{3}}$



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$$\langle \Gamma_R^2 \rangle = \int_{\mathcal{D}} d^2\mathbf{r} \int_{\mathcal{D}} d^2\mathbf{r}' \langle \omega(\mathbf{r}) \omega(\mathbf{r}') \rangle$$

$\sim$ (viscosity)^2 x (number density x R^2)^2 x $R^{-4/3}$

$\sim$ (viscosity)^2 x ($\eta_K^{-2} R^2$)^2 x ($\eta_K^{4/3} R^{-4/3}$)

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$$\langle \Gamma_R^2 \rangle = \int_{\mathcal{D}} d^2\mathbf{r} \int_{\mathcal{D}} d^2\mathbf{r}' \langle \omega(\mathbf{r}) \omega(\mathbf{r}') \rangle$$

Elementary circulations correlated as $1/r^{4/3}$

$$\mathbb{E}[\Gamma_R^2] \sim \boxed{\left(\sqrt{\frac{\epsilon}{\nu^3}} R^2\right)}^2 \boxed{\left(\frac{\eta_K}{R}\right)^{\frac{4}{3}}}$$

Number of structures

Model definitions:

$$\Gamma = \int_{\mathcal{D}} d^2\mathbf{r}\,\xi(\mathbf{r})\tilde{\Gamma}(\mathbf{r})$$

$$\xi(\mathbf{r}) = \sqrt{\epsilon(\mathbf{r})}$$

$$\mathbb{E}[\epsilon(\mathbf{r})\epsilon(\mathbf{r}')] \sim |\mathbf{r} - \mathbf{r}'|^{-\mu}$$

Two-Dimensional Fractional Brownian Motion:

$$\tilde{\Gamma}(\mathbf{r}) = \int d^2\mathbf{k}\,\psi(\mathbf{k})k^{\frac{\alpha}{2}-1}\exp\left(i\mathbf{k}\cdot\mathbf{r} - k\frac{\eta_K}{2}\right)$$

$\alpha = 4/3 - \mu/2$
$\mu = 0.17$

Complex Random Gaussian Field
with vanishing mean and correlator

$$\langle\psi(\mathbf{k}_1)\psi(\mathbf{k}_2)\rangle = \delta^2(\mathbf{k}_1 - \mathbf{k}_2)$$

Model definitions

$$\Gamma = \xi_{\rm cg}(\mathcal{D}) \int_{\mathcal{D}} d^2\mathbf{r}\tilde{\Gamma}(\mathbf{r})$$

$$\xi_{\rm cg}(\mathcal{D}) \equiv \frac{1}{A}\int_{\mathcal{D}} d^2\mathbf{r}\xi(\mathbf{r})$$

This integration is actually a summation over a point process.

2D vortices have radii = $a\eta_K$

Extension of the OK62 Phenomenology

$$\xi_R \equiv \frac{1}{\pi R^2}\int_{\mathcal{D}} d^2\mathbf{r}\,\xi(\mathbf{r}) = \frac{\xi_0}{\pi R^2}\int_{\mathcal{D}} d^2\mathbf{r}\sqrt{\frac{\epsilon(\mathbf{r})}{\epsilon_0}}$$

$$\xi_R = \xi_0 \exp(-X_R)$$

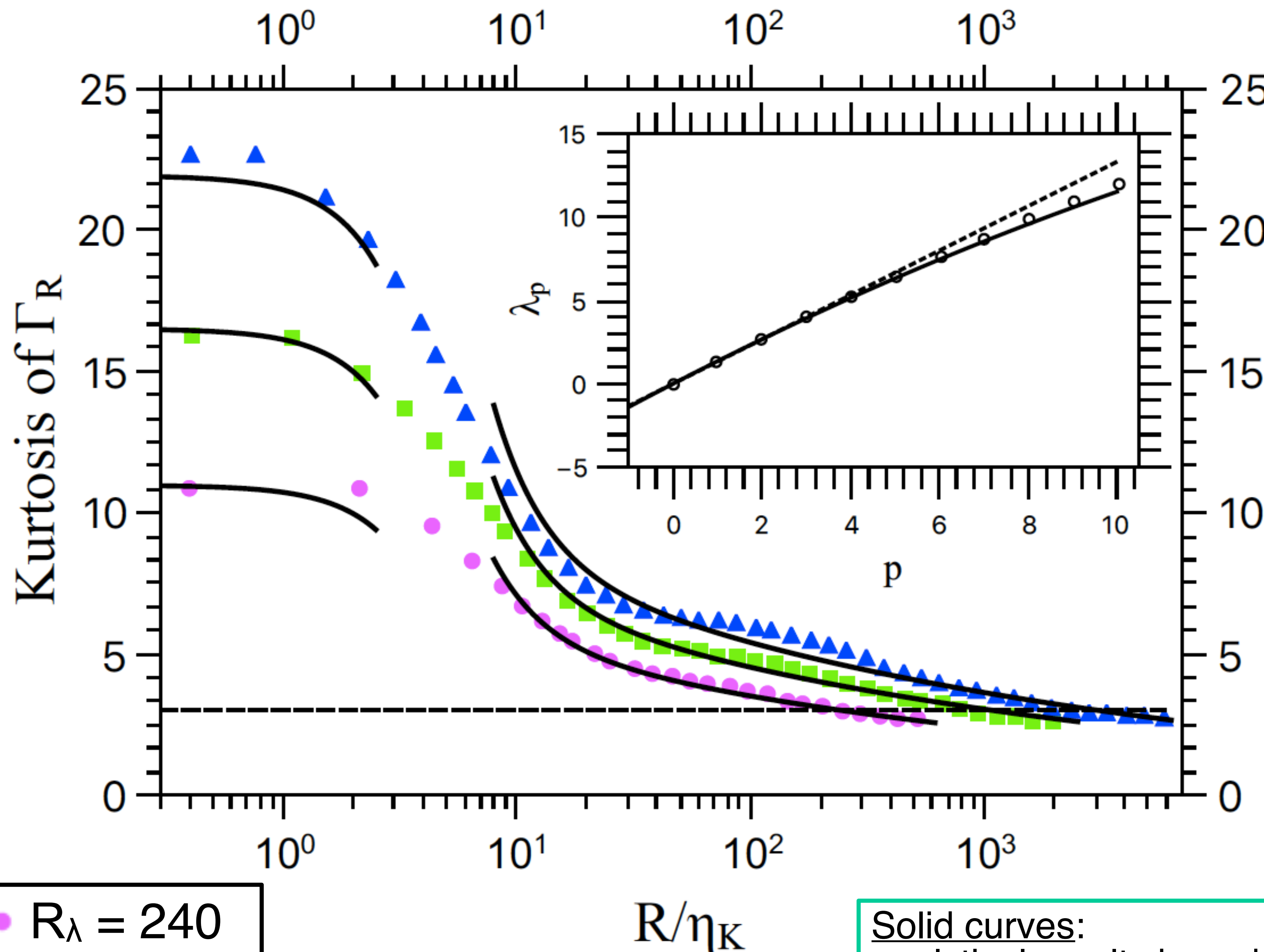
X_R is a Gaussian random variable with mean and variance equal to

$$\bar{X}_R = \frac{3\mu}{8}\ln\left[\frac{R_\lambda}{\sqrt{15}}\left(\frac{\eta_K}{bR+\eta_K}\right)^{\frac{2}{3}}\right]$$

$a \approx 3.3$

$b \approx 1.6 - 1.8$

Model parameters: a, b, and σ = planar vortex density



● $R_\lambda = 240$
■ $R_\lambda = 650$
▲ $R_\lambda = 1300$

← Iyer et al. PRX (2019)

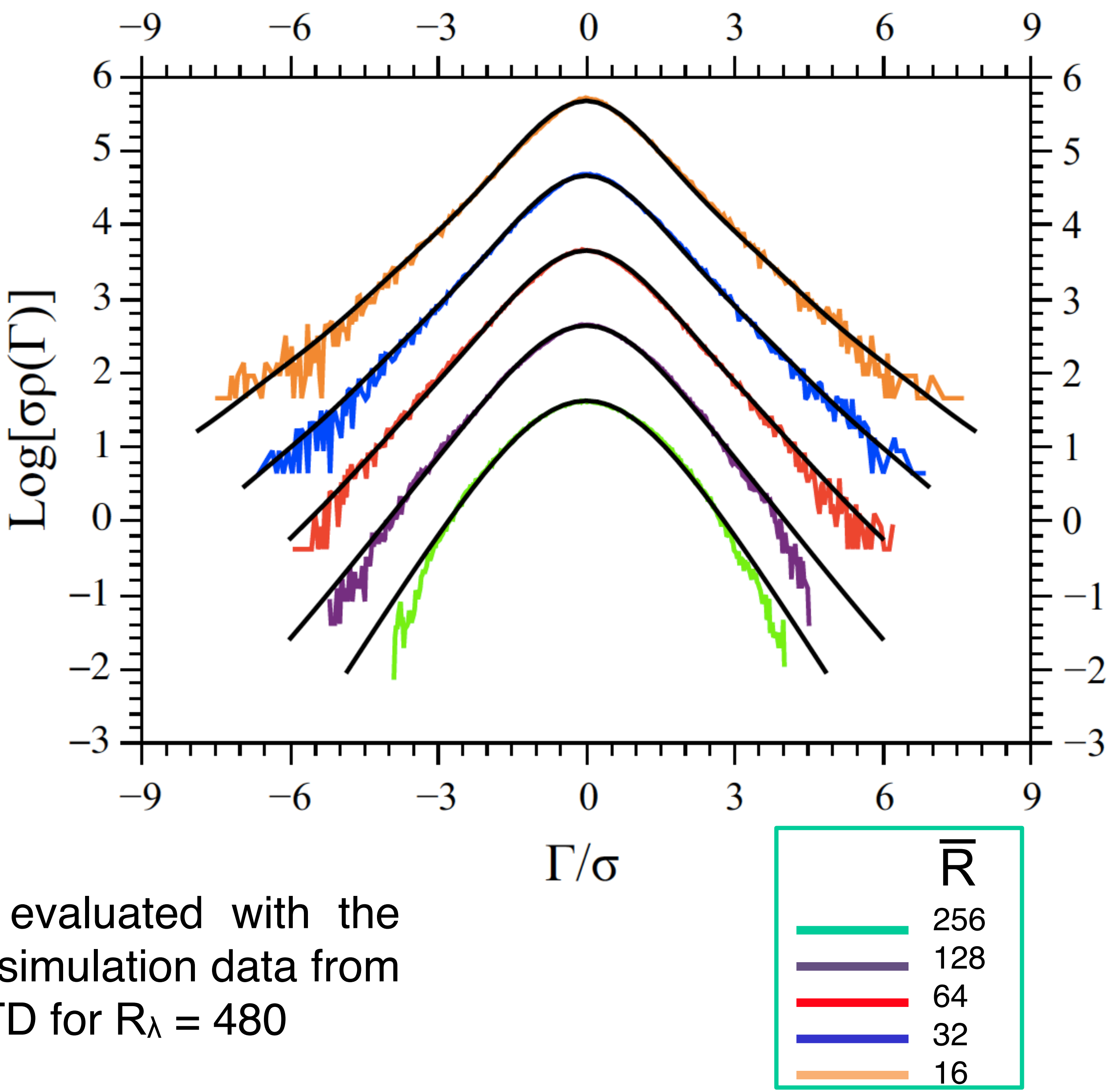
<u>Solid curves</u>:
analytical results based on an underlying Poisson Process

Circulation Probability Distribution Functions (CPDFs)

$$
\begin{aligned}
Z(\zeta) &= \langle\langle\langle \exp\left[i\zeta\Gamma_R\right]\rangle\rangle\rangle_{(\tilde{\omega},\xi_R,\sigma)} \\
&= \int_0^\infty d\xi\, f_R(\xi)\langle \exp\left[-\frac{1}{2}\zeta^2\xi^2\Omega\right]\rangle_\sigma
\end{aligned}
$$

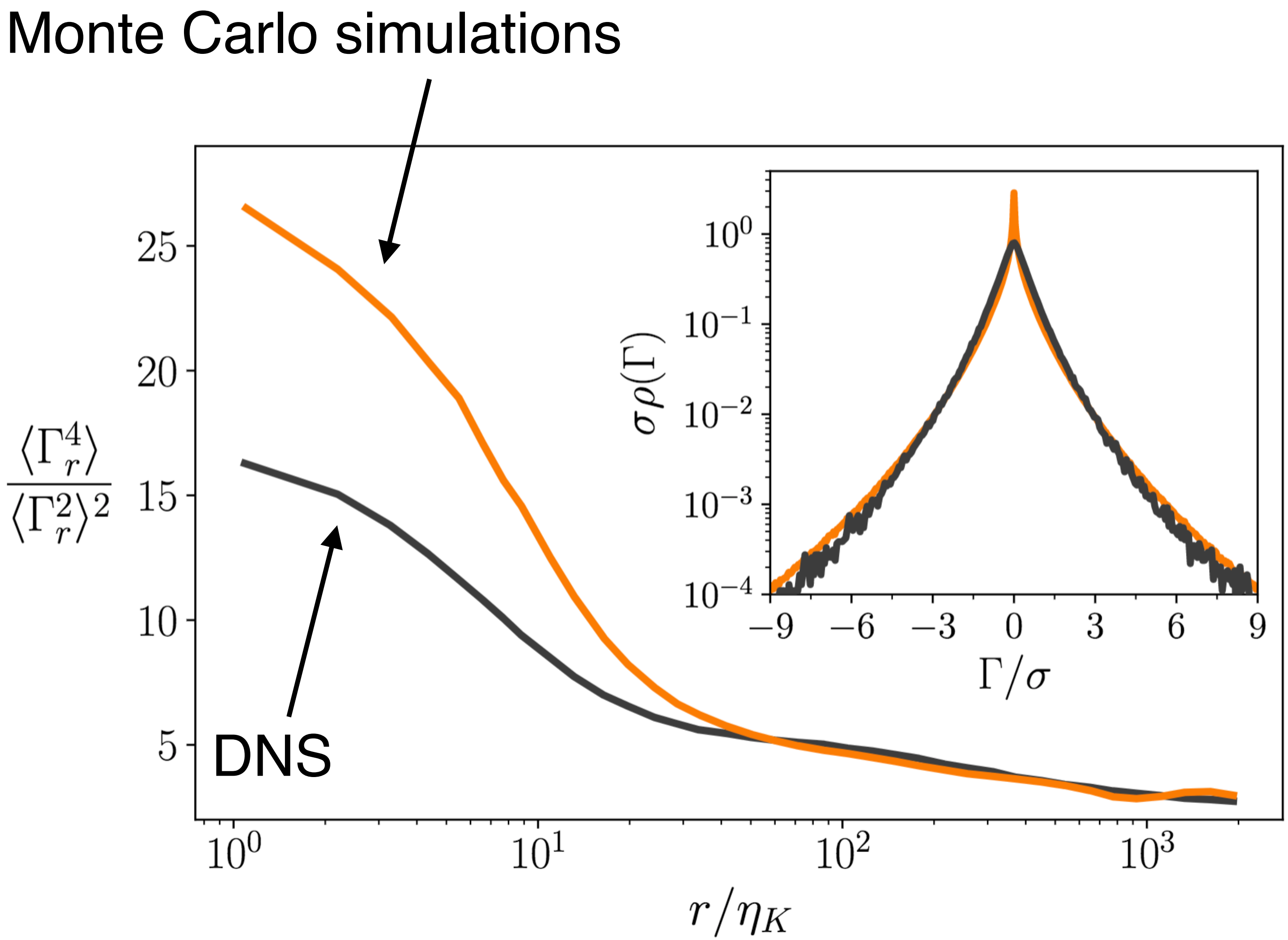
$$
\Omega \equiv \int_{\mathcal{D}} d^2\mathbf{r} \int_{\mathcal{D}} d^2\mathbf{r}' \,\langle\tilde{\omega}(\mathbf{r})\tilde{\omega}(\mathbf{r}')\rangle\, \sigma(\mathbf{r})\sigma(\mathbf{r}')
$$

$$
\rho_R(\Gamma) = \frac{1}{\sqrt{2\pi\bar{\Omega}}}\int_0^\infty d\xi \frac{1}{\xi} f_R(\xi)\exp\left(-\frac{\Gamma^2}{2\xi^2\bar{\Omega}}\right)
$$



cPDFs evaluated with the help of simulation data from the JHTD for $R_\lambda = 480$

A drawback?



L.M., R.M. Pereira, and V.J. Valadão, PRE Letter (2022).

Revisiting the Vortex Gas Model

* Are vortices really distributed as a 2D Poisson point process?

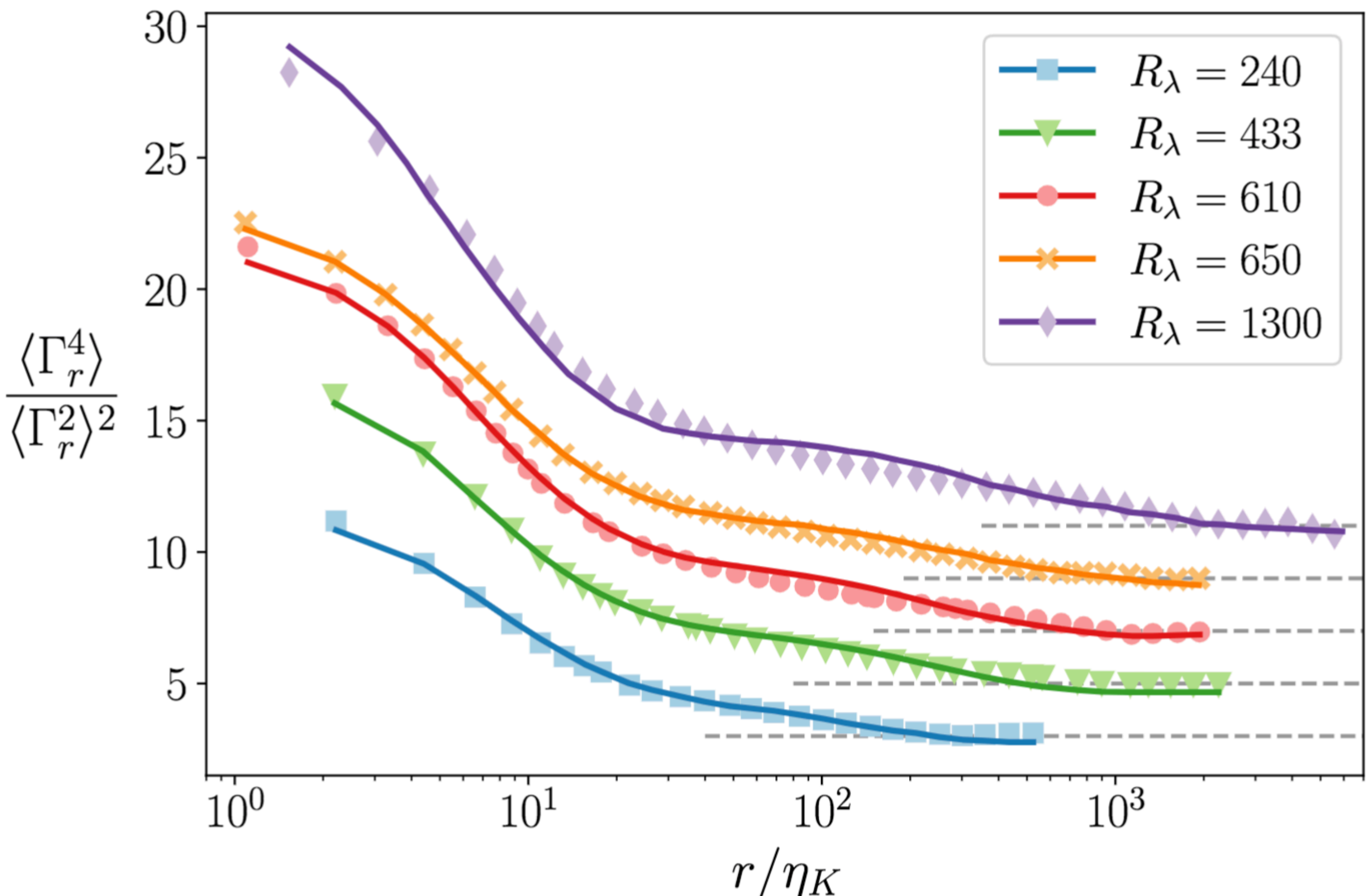
* We have, in fact, neglected subdominant terms in the perturbative evaluation of circulation statistical moments at small scales, but they do not fix the problem…

* ….however, subdominant terms may be suppressed for a model of mutually repelling vortex structures;

* Multifractality breaking (see ahead) indicates that the surface vortex density is an upper-bounded field (likely to be related, in a phenomenological way, to the mutual repulsion between vortex structures)

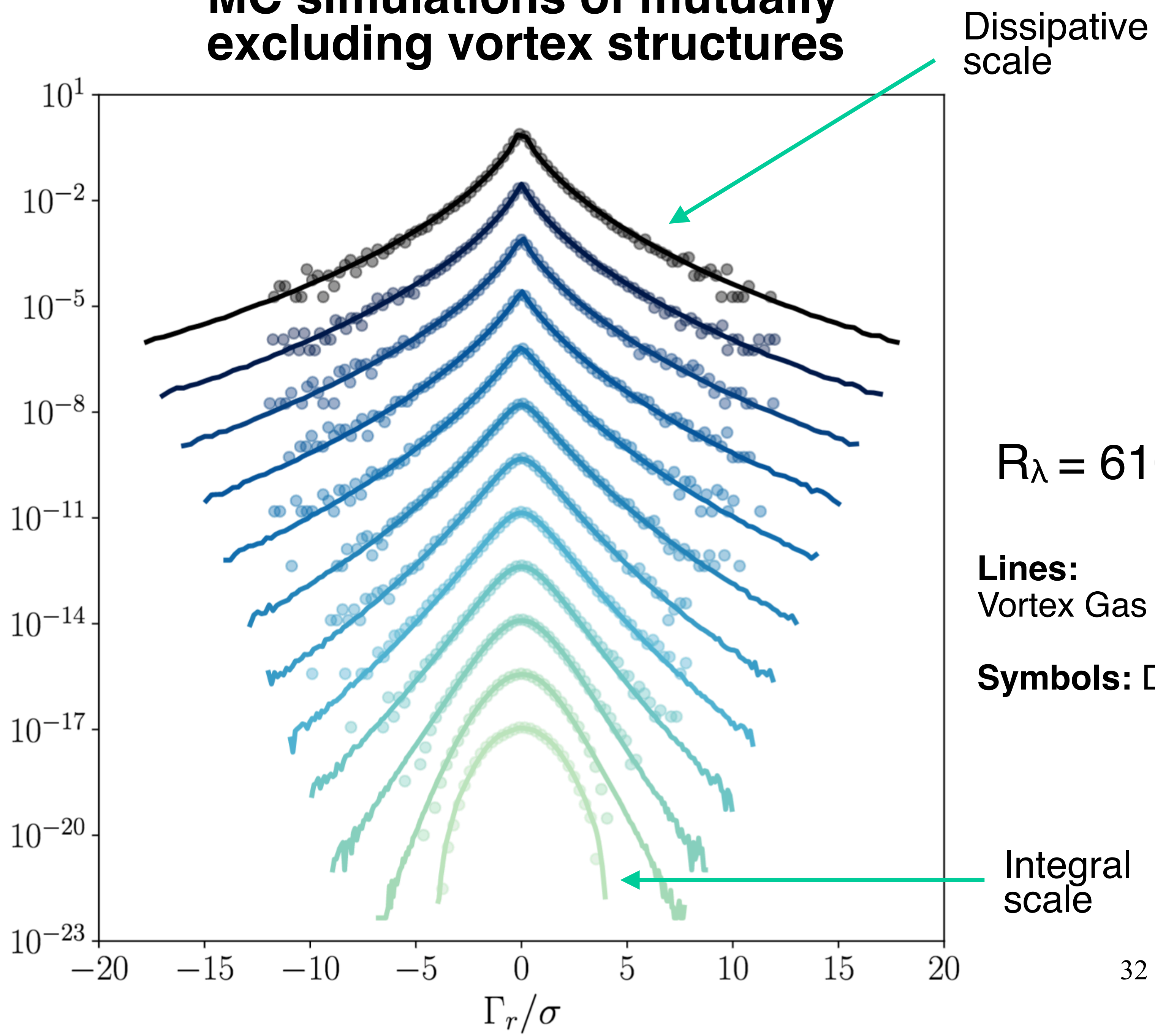
Fast Monte Carlo simulations of a system of hard disks (mutually repelling vortex structures in our case) have been a challenge for decades. A solution by Bernard et al., PRE **80**, 056704 (2009).

MC simulations of mutually excluding vortex structures



Symbols: Vortex Gas Model
Lines: DNS

MC simulations of mutually excluding vortex structures



$R_\lambda = 610$

Lines: Vortex Gas Model

Symbols: DNS

The distribution of Vortex Structures as a Hard Disk Point Process (validation)

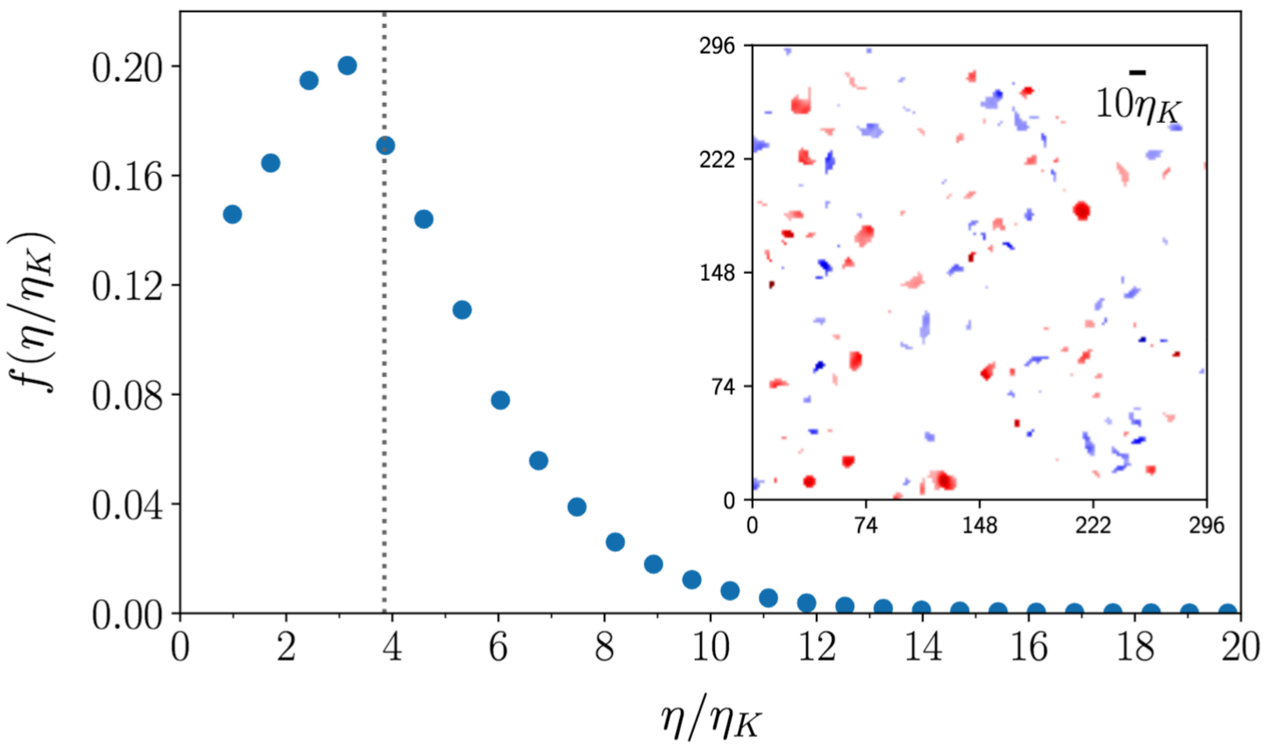


FIG. 4: Distribution of estimated radii in DNS detected structures ($R_\lambda = 610$). Vertical dotted line: the radius mean value $\langle\eta\rangle = 3.85\eta_K$. Inset: a snapshot of the planar vortex spots (value axes labels given in units of η_K). Red and blue spots denote positive and negative vorticity, respectively.

The distribution of Vortex Structures as a Hard Disk Point Process (validation)

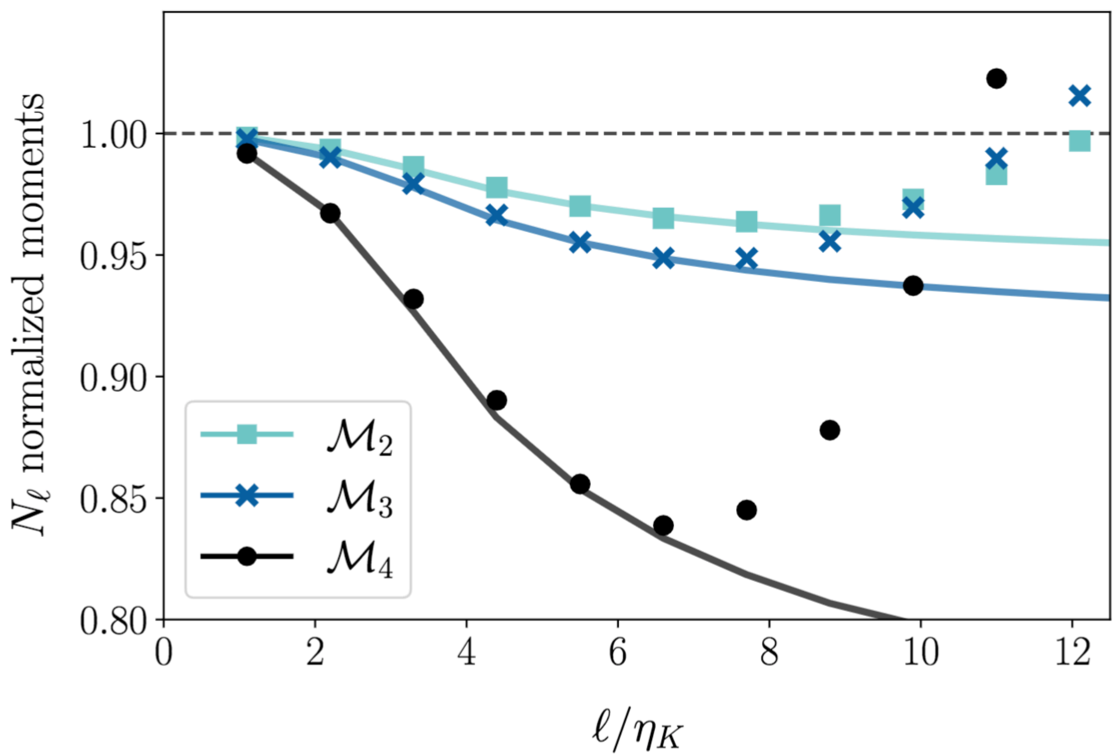


FIG. 5: Normalized statistical moments of the number of points N_ℓ inside squares of side ℓ as a function of ℓ/η_K. Symbols: DNS detected structures ($R_\lambda = 610$). Solid lines: hard disk point process with the same mean number of points and mean radii $\langle\eta\rangle = 3.8\eta_K$. Dashed: Poisson point process.

IV. Multifractality Breaking

Linearization Effect - Field Theoretical Description

$$\Gamma = \xi_{\rm cg}(\mathcal{D}) \int_{\mathcal{D}} d^2\mathbf{r}\tilde{\Gamma}(\mathbf{r})$$

This is the coarse-grained field that we want to model in an improved way.

Multifractality from *Gaussian Multiplicative Chaos* (GMC)

Rhodes and Vargas, Probab. Surv. (2014)

$$\psi_a(x) \equiv \frac{1}{a^d} \int_{\mathcal{D}_a} d^d x' \psi(x')$$

$$\psi(x) = \psi_0 \exp\left\{\gamma\phi(x) - \frac{\gamma^2}{2}\mathbb{E}[\phi^2]\right\}$$

$$d\mu[\phi] = D[\phi] \exp\{-S[\phi]\}$$

$$S[\phi] = \frac{1}{2} \int_{\mathcal{D}_L} d^2 x (\partial_i \phi)^2$$

→

$$\mathbb{E}[(\psi_a(x))^q] = c_q \psi_0^q \left(\frac{a}{L}\right)^{\tau_q}$$

$$\mathbb{E}[(\psi(x))^q] = \left(\frac{\eta}{L}\right)^{\tau_q}$$

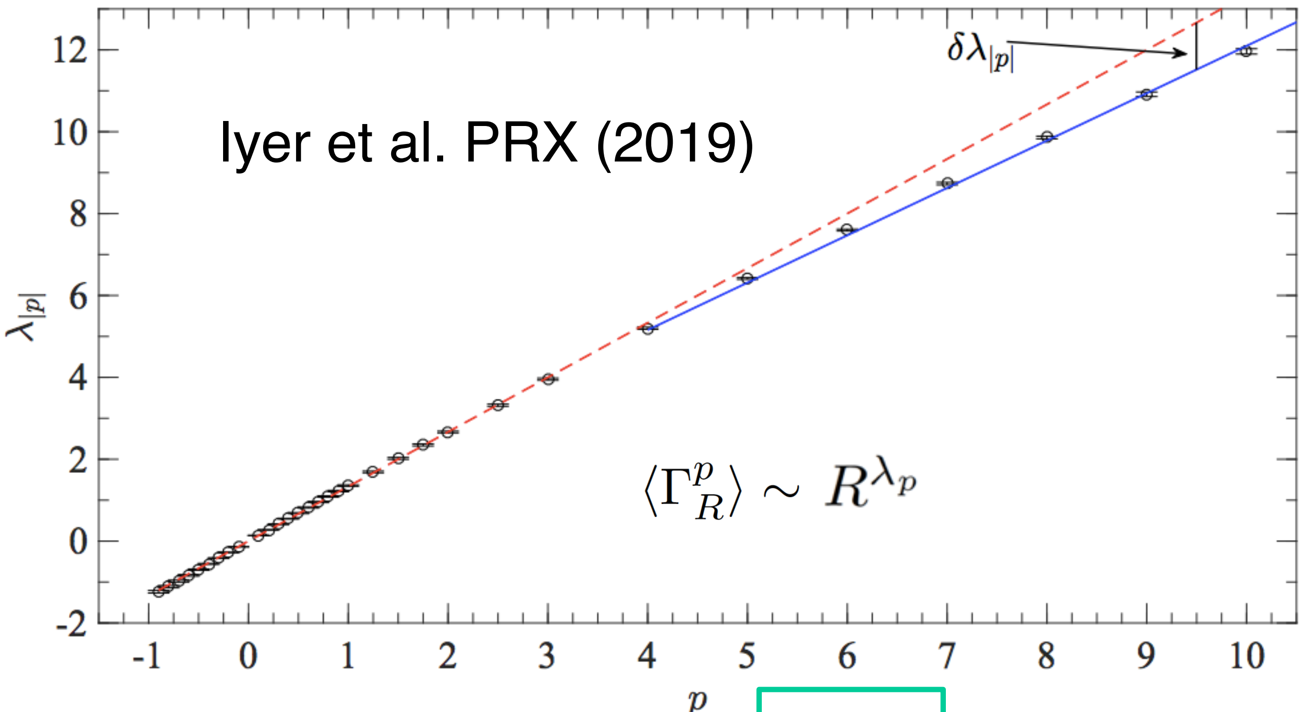
$$\tau_q = \frac{\gamma^2}{4\pi} q(1-q)$$

L.M., PRE (2021)

Linearization Effect

At enough high orders, scaling exponents of the statistical moments of physical observables of interest are found to depend linearly upon its moment orders, at variance with the typical non-linear profiles predicted for multifractal systems. Molchan, CMP (1996), Lashermes et al. IJWMIP (2004).

This mechanism could explain the peculiar scaling features of circulation observed in numerical simulations:



We propose, as a way to account for the linearization effect, in the language of the GMC theory, the following alternative scenario (see the paper for details)

$$\psi(x) = \frac{\psi_0}{\mathbb{E}[\tilde{\psi}(x)]}\tilde{\psi}(x) \qquad \tilde{\psi}(x) = \exp[\gamma\phi(x)]$$

$$\psi_a(x) \equiv \frac{1}{a^d}\int_{\mathcal{D}_a} d^d x' \psi(x')$$

where, now,

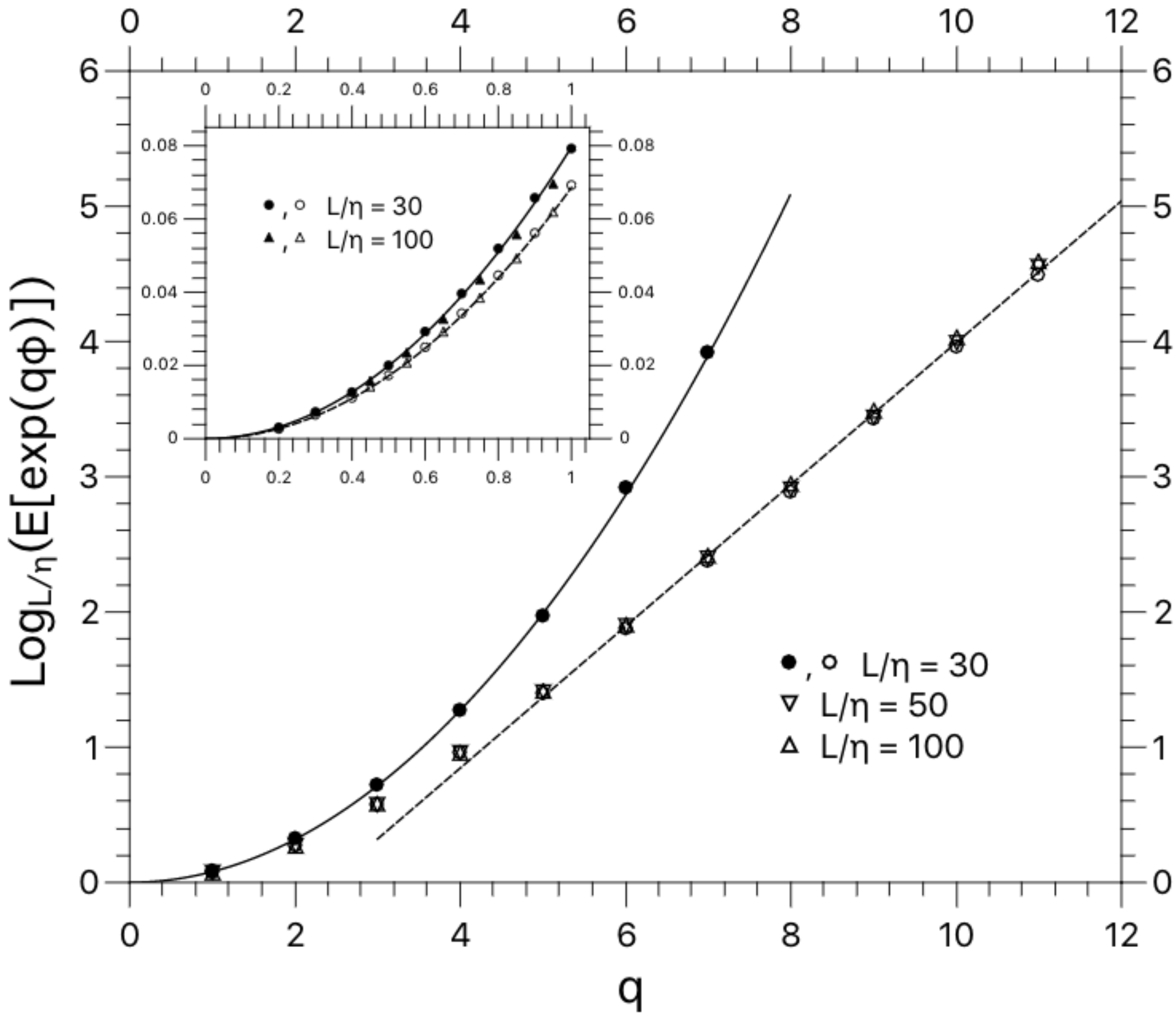
$$S[\phi] = \int d^2x \left[\frac{1}{2}(\partial_i \phi)^2 + V(\phi)\right] \ , \qquad V(\phi) = \begin{cases} 0 \ , \ \text{if } \phi < \phi_0 \ , \\ V_0 \ , \ \text{if } \phi \geq \phi_0 \ , \end{cases}$$

with

$V_0 \to \infty$ and $\phi_0 = C \ln(L/\eta)$

$$\mathbb{E}[(\psi_a(x))^q] = \ ?$$

Results
(Monte Carlo Simulations)



Results
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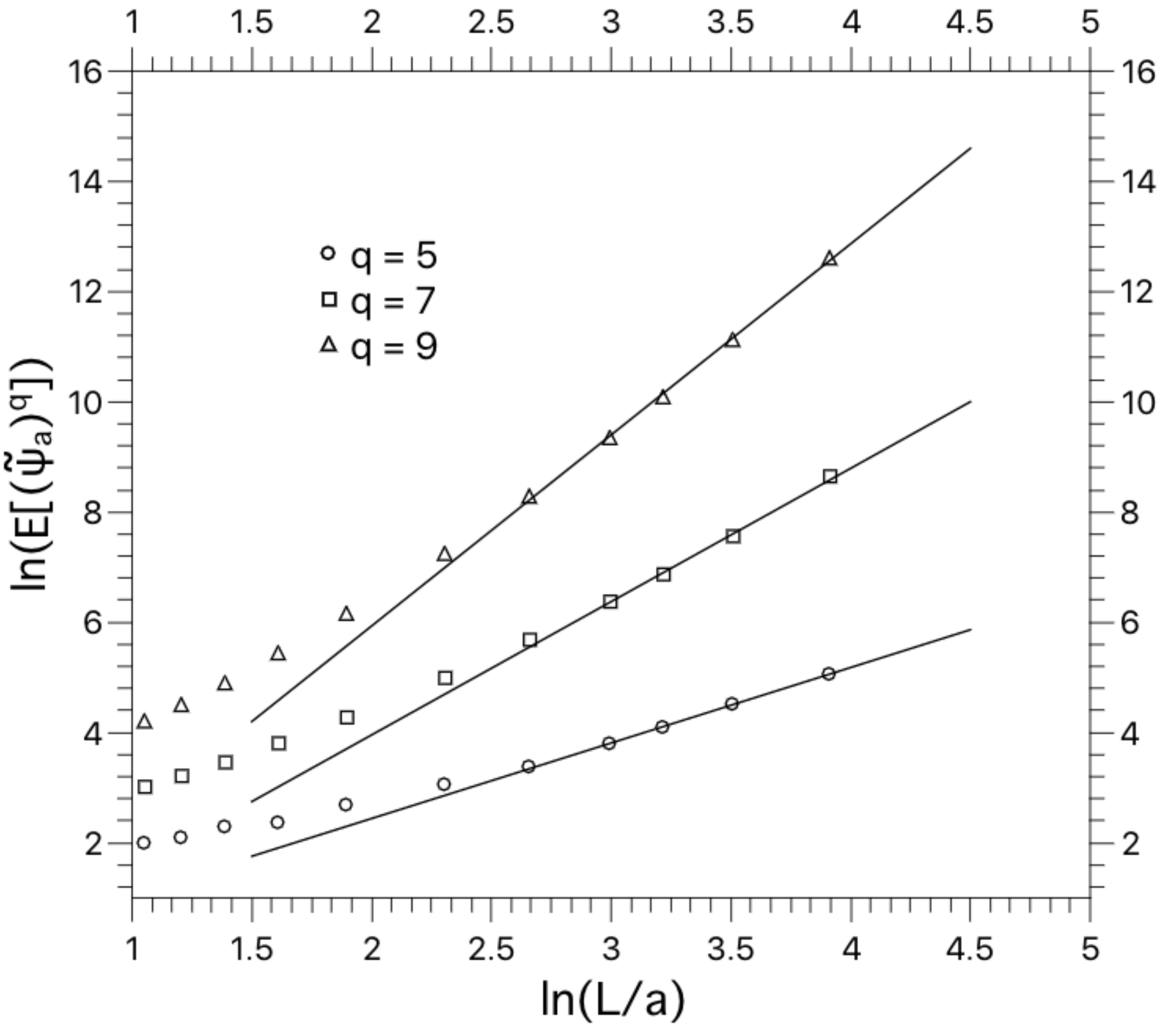
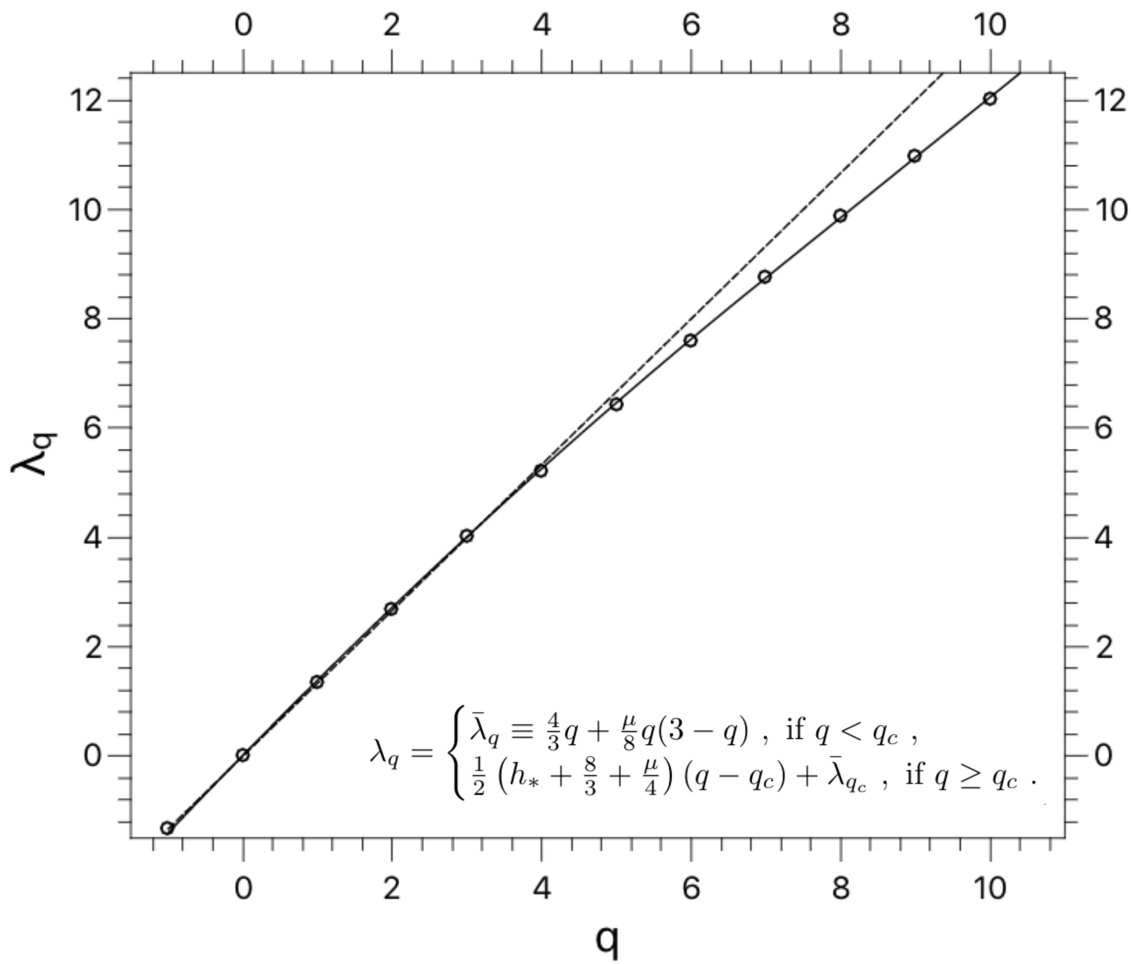


Results

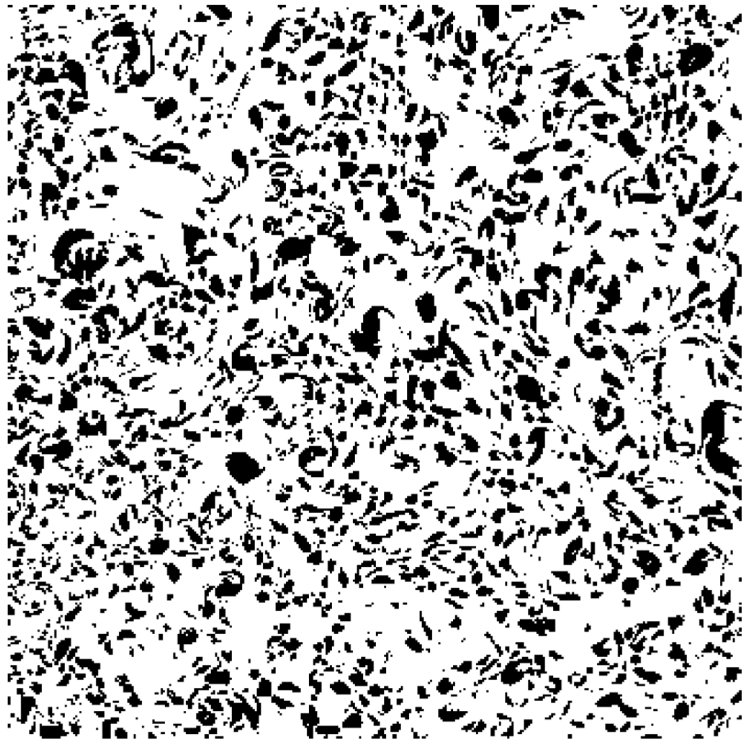
(Application to Circulation Statistics)



$h_* = -0.5$, $\mu = 0.17$, and $q_c = 6.88$ are phenomenological parameters obtained from previous experimental and numerical studies:

Meneveau and Sreenivasan PRL (1987), Chhabra et al. PRA (1989), Tang et al. JFM (2020).

Vortex Surface Density and GMC

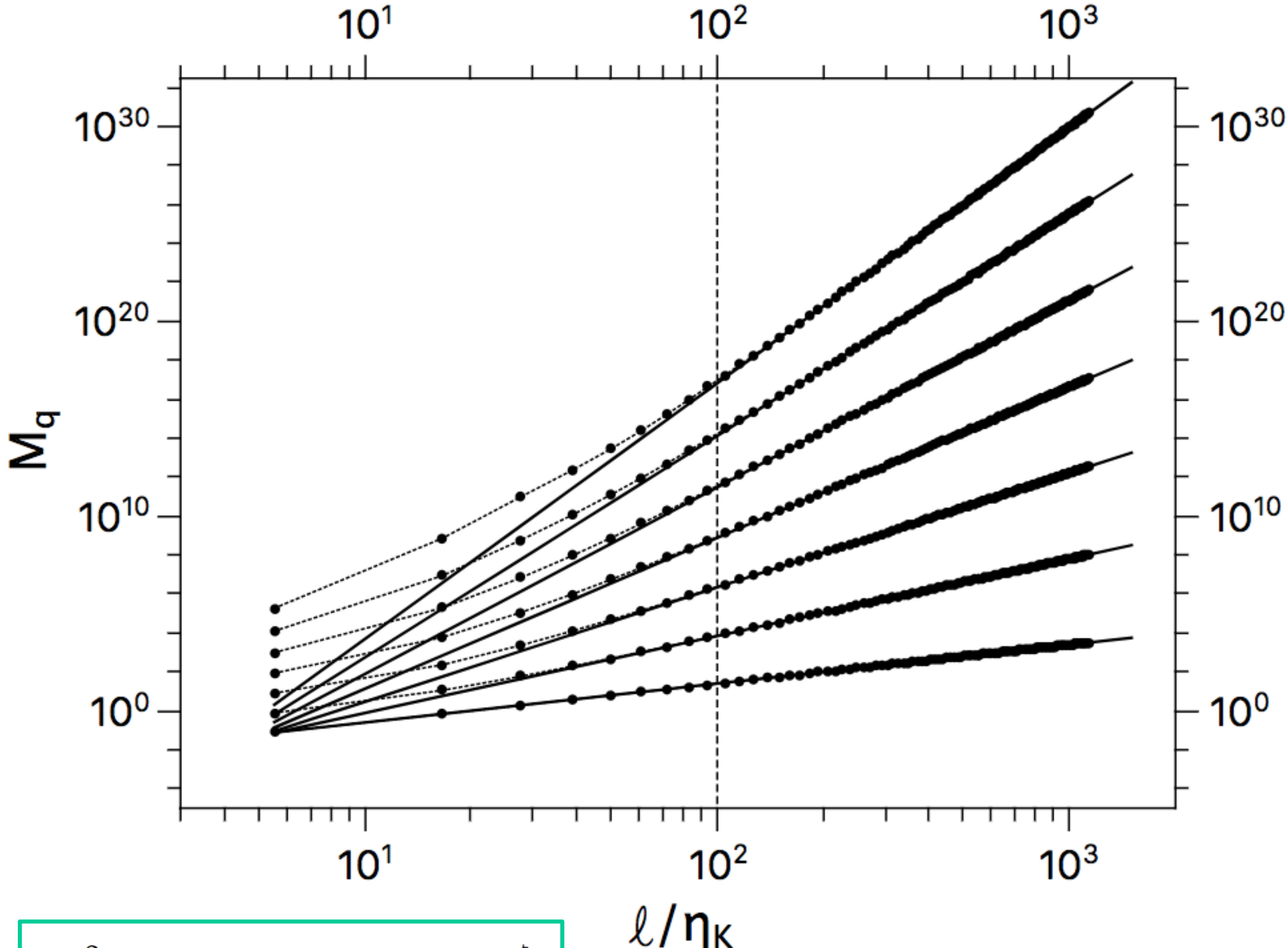


$$\mathbb{E}[\xi_{\mathrm{cg}}^q(\mathcal{D}_\ell)] \sim \ell^{\zeta_q}$$

$$\zeta_q = \frac{\mu}{8} q(1-q)$$

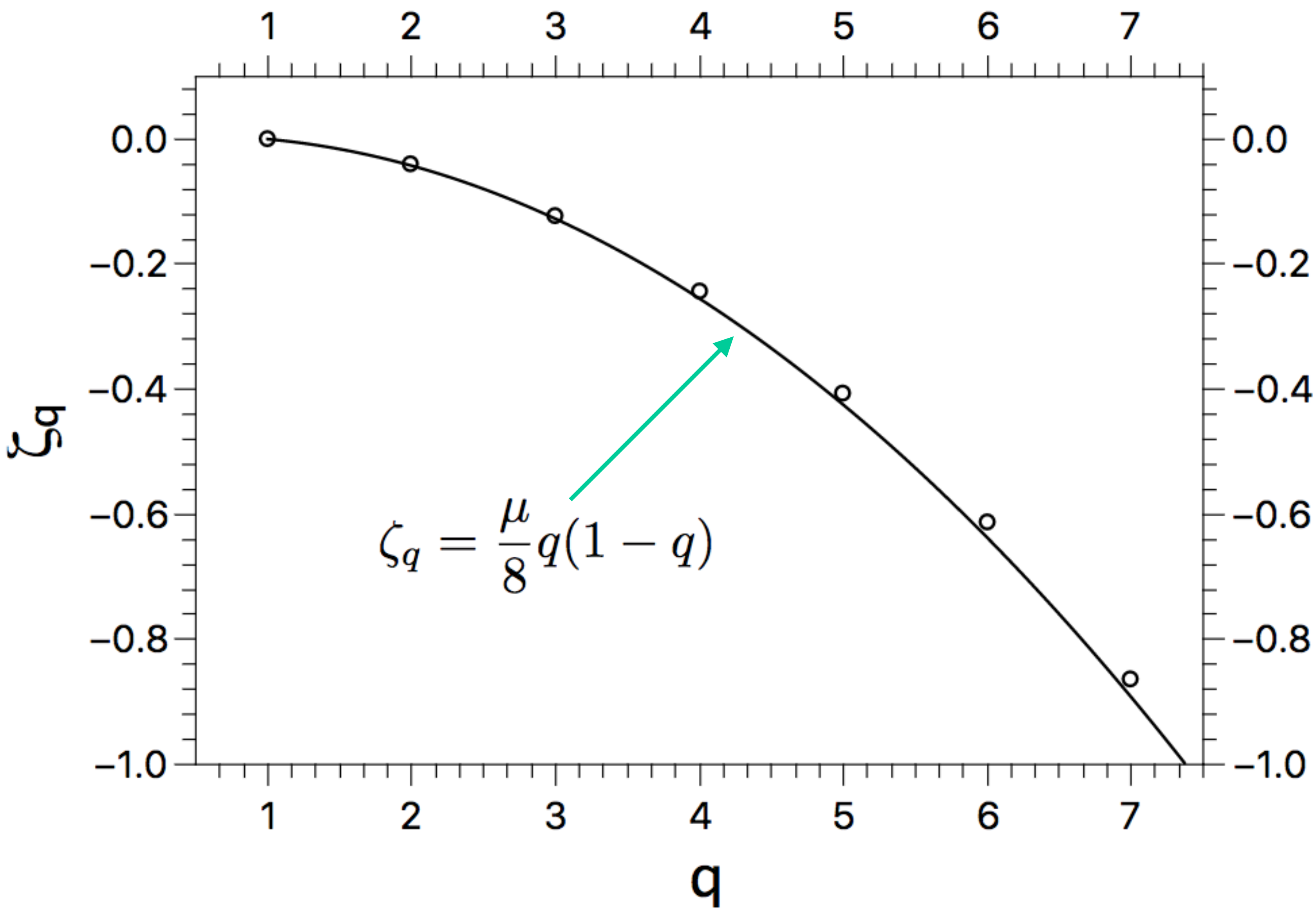
$$M_q(\ell) \equiv \mathbb{E}[N_\ell^q]$$

$$\ell^{-2q} M_q(\ell) \sim \mathbb{E}[\xi_{\mathrm{cg}}^q(\mathcal{D}_\ell)] \sim \ell^{\zeta_q}$$



$$\ell^{-2q} M_q(\ell) \sim \mathbb{E}[\xi_{\mathrm{cg}}^q(\mathcal{D}_\ell)] \sim \ell^{\zeta_q}$$

L.M. and R.M. Pereira, ArXiv (2022)

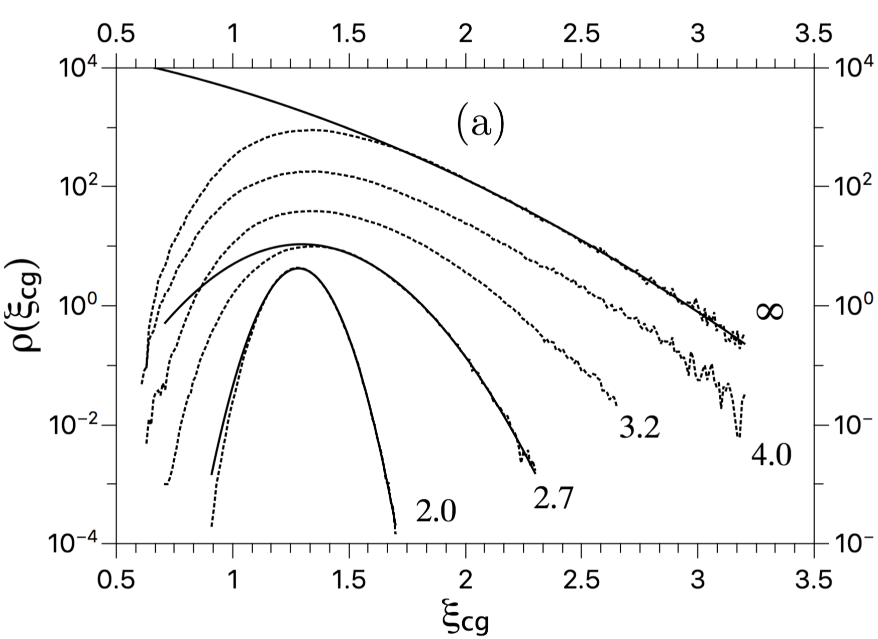


1
2
3
4
5
6
7
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
ζ_q
$\zeta_q = \frac{\mu}{8} q(1-q)$
q

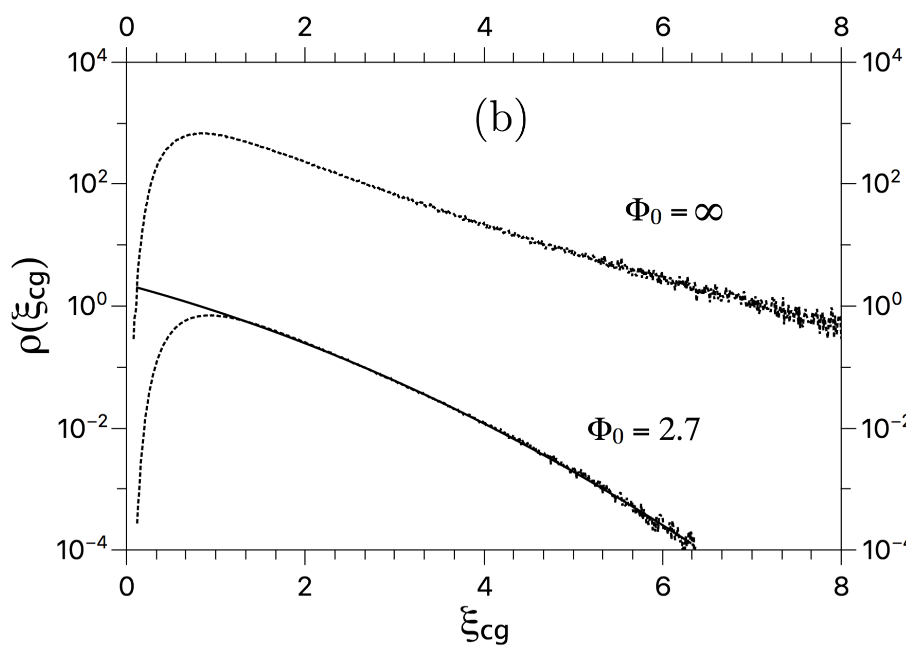
V. Extreme Circulation Events

Bounded fluctuations of $\xi(r)$ may lead to Gaussian ξ_{cg}

We have investigated this conjecture, through Lattice Monte Carlo simulations (lattice dimensions 100 x 100).



subdomain dims: 50 x 50



subdomain dims: 5 x 5

L.M. and R.M. Pereira, ArXiv (2022)

Recall that

$$\Gamma = \xi_{\rm cg}(\mathcal{D}) \int_{\mathcal{D}} d^2\mathbf{r}\tilde{\Gamma}(\mathbf{r})$$

If, now,

$$\xi_{\rm cg}(\mathcal{D}) = \mathbb{E}[\xi_{\rm cg}(\mathcal{D})] + \tilde{\xi}_{\rm cg}(\mathcal{D})$$

$$\Downarrow$$

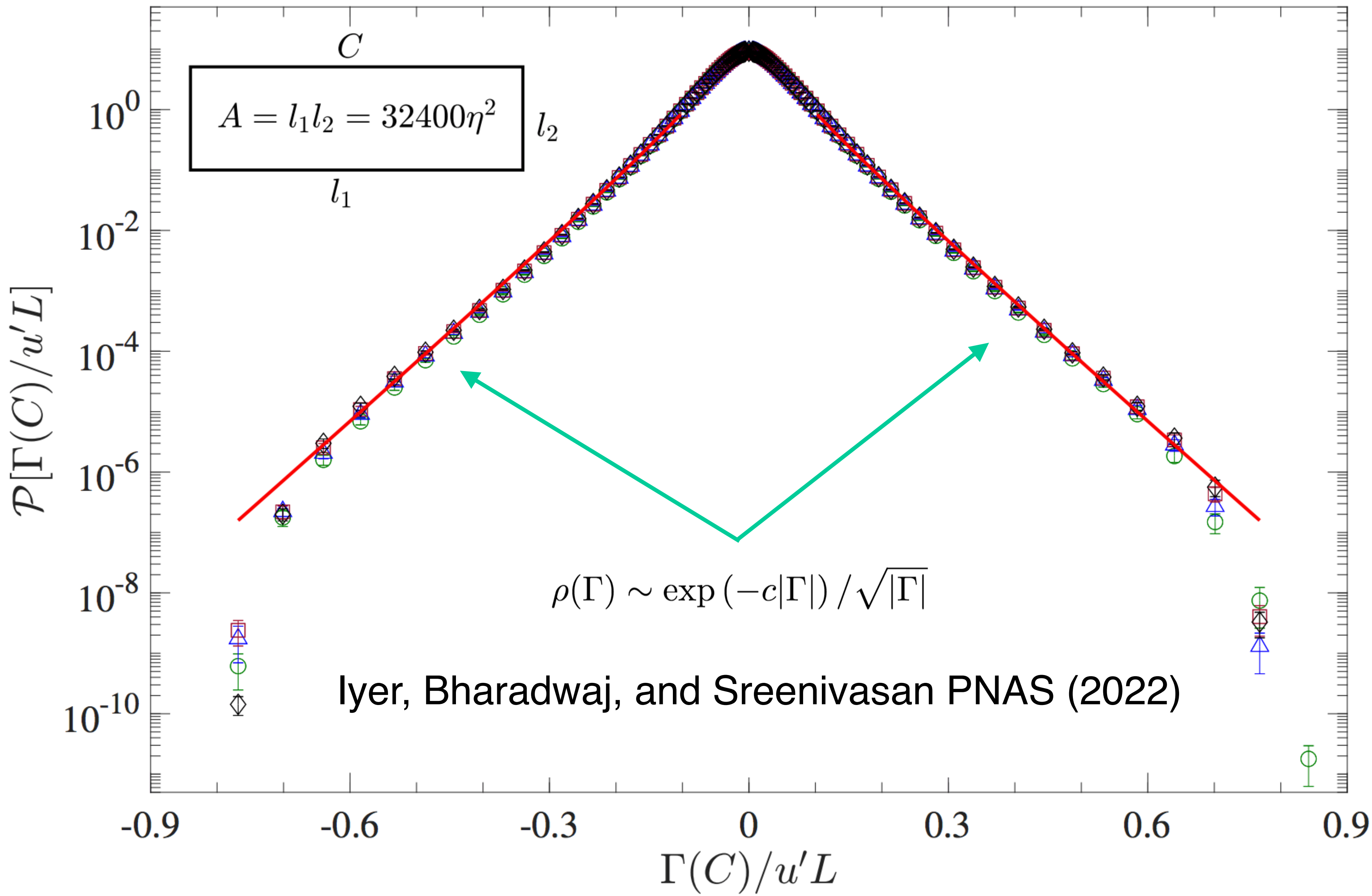
$$\Gamma = \mathbb{E}[\xi_{\rm cg}(\mathcal{D})] \int_{\mathcal{D}} d^2\mathbf{r}\tilde{\Gamma}(\mathbf{r}) + \tilde{\xi}_{\rm cg}(\mathcal{D}) \int_{\mathcal{D}} d^2\mathbf{r}\tilde{\Gamma}(\mathbf{r})$$

$$\xi_{\rm cg}(\mathcal{D}) = \mathbb{E}[\xi_{\rm cg}(\mathcal{D})] + \tilde{\xi}_{\rm cg}(\mathcal{D})$$

$$\Gamma = \mathbb{E}[\xi_{\rm cg}(\mathcal{D})] \int_{\mathcal{D}} d^2\mathbf{r}\tilde{\Gamma}(\mathbf{r}) + \tilde{\xi}_{\rm cg}(\mathcal{D}) \int_{\mathcal{D}} d^2\mathbf{r}\tilde{\Gamma}(\mathbf{r})$$

product of two Gaussians

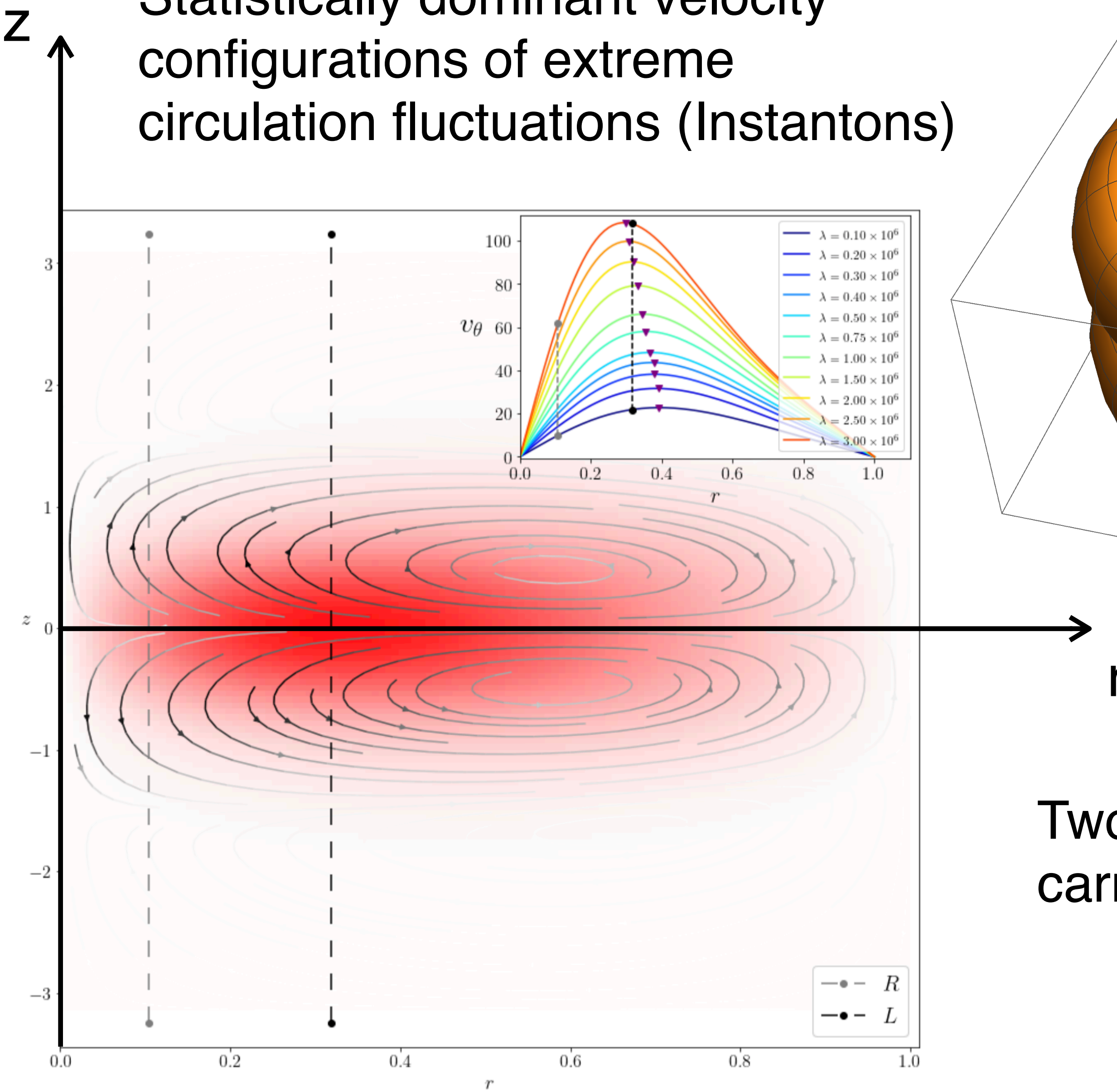
$$\rho(\Gamma) \sim \exp\left(-c|\Gamma|\right) / \sqrt{|\Gamma|}$$ (at the tails)

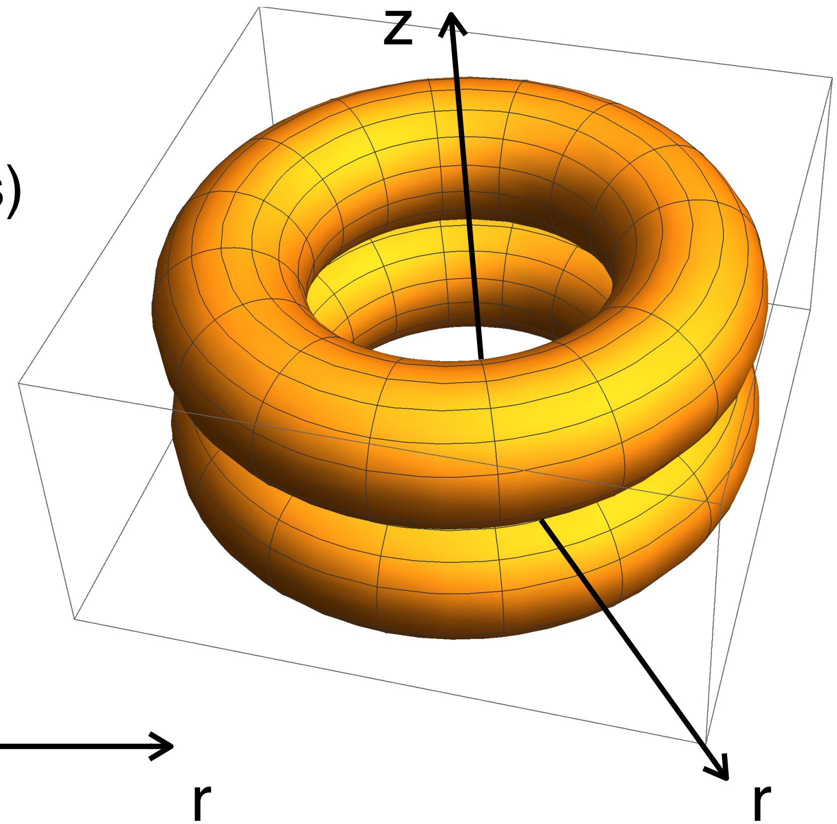


Iyer, Bharadwaj, and Sreenivasan PNAS (2022)

See also Migdal, IJMPA (2020)

Statistically dominant velocity configurations of extreme circulation fluctuations (Instantons)





Two vortex rings which carry opposite helicity

G.B. Apolinário, L.M, R.M. Pereira, and V.J. Valadão, PLA (2022)

VI. Minimal Surfaces & Circulation Statistics

See Iyer, Bharadwaj, and Sreenivasan PNAS (2022); L.M. PNAS (2022)

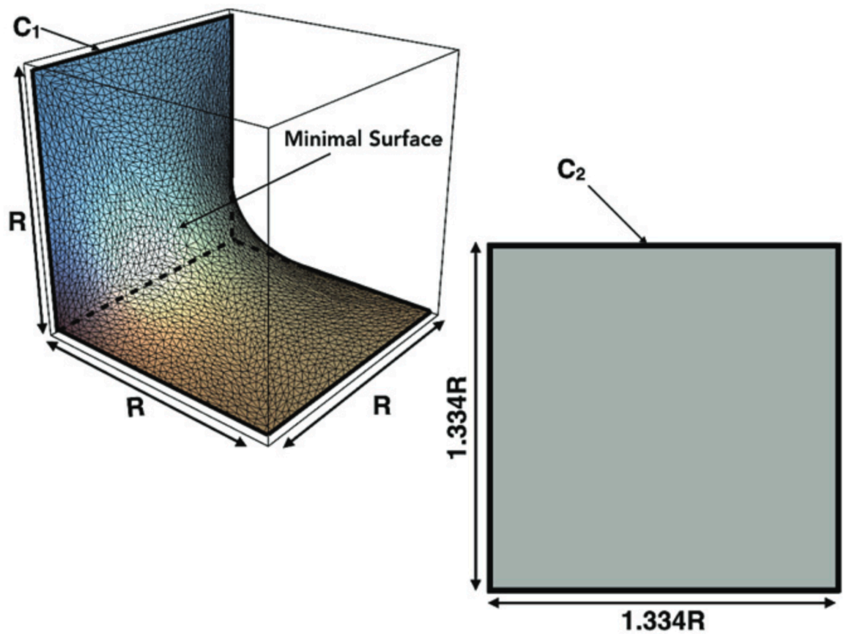
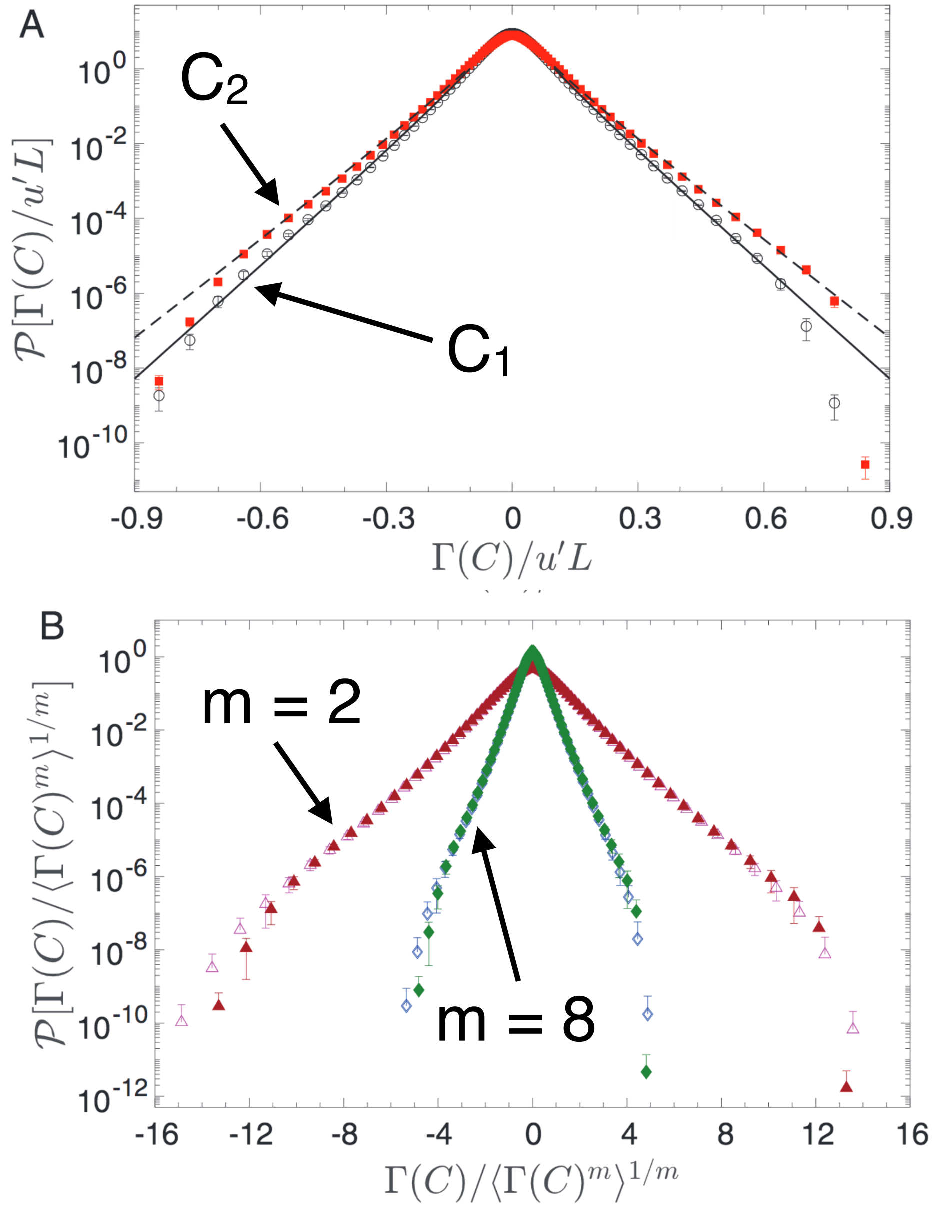
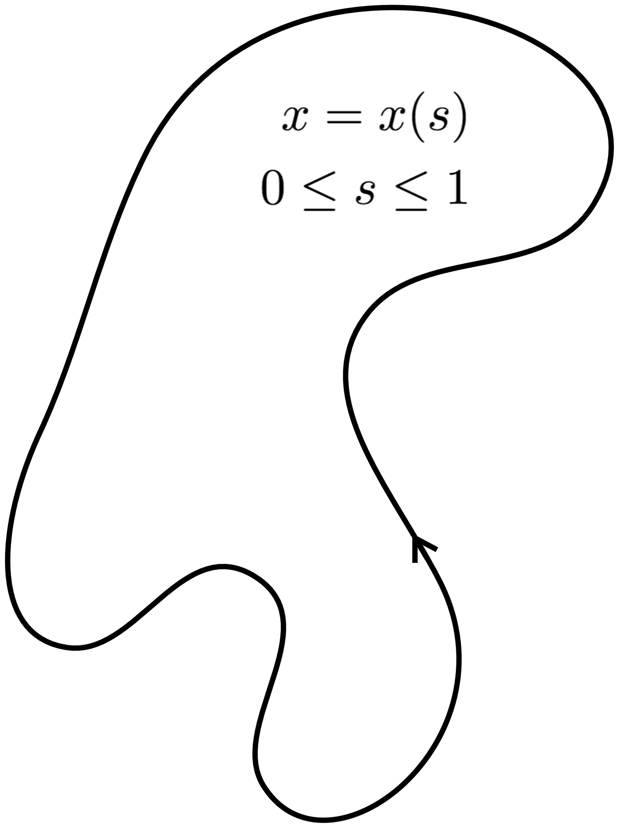


Fig. 4. Contour C_1 is a nonplanar contour with side R and minimal area $A_{C_1} = 1.78R^2$. Contour C_2 is a square with the same planar area $A_{C_2} = 1.78R^2$.



Minimal Surface Argument



$$\rho(\Gamma) \sim \exp(-c\Gamma)$$

$$\Gamma_0 \equiv \sqrt{\mathbb{E}[\Gamma^2]}$$

$$\rho(\Gamma) \sim \exp(-c\Gamma_0\Gamma/\Gamma_0)$$

$$c\Gamma_0 = F[x(s)/L; \mathrm{Re}]$$

$$F[x_1(s)/L; \mathrm{Re}] = F[x_2(s)/L; \mathrm{Re}]$$

$$\Rightarrow \quad F[x(s)/L; \mathrm{Re}] = f(A_{min}/L^2; \mathrm{Re})$$

VII. Conclusions & Outlook

The combination of multiplicative cascade and structural approaches is the main feature of a **vortex gas model** which allows us to obtain key statistical properties of turbulent circulation.

To implement / improve / investigate:

- Extension to non-planar loops; minimal surfaces

- Kurtosis plateau at higher Reynolds numbers (?)

- 3D reconstruction of vortex tubes from the 2D slices

- Instanton filtering for the extreme circulation events

- Field theoretical analysis of multifractality breaking